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A Model for Teaching Mathematical Argument at the Elementary Grades

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ABSTRACT

Over years of collaborating with elementary-school teachers to research students' thinking about the "big ideas" of K-6 mathematics, particular attention was given to generalizations about the operations—addition, subtraction, multiplication, and division—and arguments that explain why these generalizations are true. Through this work, we created a model of five phases that separate different points of focus in the complex process of formulating and proving such generalizations: 1) noticing patterns, 2) articulating conjectures, 3) representing with specific examples, 4) creating representation-based arguments, and 5) comparing and contrasting operations. In this paper, we illustrate the phases with classroom examples as students investigate a set of generalizations. We then present assessment results from classrooms of project teachers who engaged their students in this content.

Key Words: Argument, Generalization, Conjecture, Representation, Operations

A MODEL FOR TEACHING MATHEMATICAL ARGUMENT AT THE ELEMENTARY GRADES

In our collaborative work with teachers who were opening up their math teaching to students' ideas (Schifter et al., 1999), we noticed over many years and across many classrooms that students frequently commented on patterns they saw in the number system. Kindergartners or first graders working on addition would remark with delight that, when you switch the order of numbers being added, the sum stays the same. They might even name this phenomenon "turn arounds" or "switch arounds." Third graders would say something similar as they solved multiplication problems: when you change the order of two factors, the product stays the same. Students of any age might exclaim that when you add two even numbers, the sum is even, but if you add an even and an odd, you get an odd number. We recognized the mathematical significance of such observations, as well as the excited energy of students who made them, but in general, while students might implicitly use such ideas to help them solve problems and teachers might briefly acknowledge students' comments, these generalizations were not raised to an explicit level as objects of study. Rather, the class moved past them and returned to the point of the lesson: finding the correct answers to the set of problems they were solving. Even teachers who were working hard to listen to and respect student ideas, who were encouraging students to make sense of computation through developing and justifying their own strategies, did not know how to follow up on such general statements.

We began to ask ourselves and our teacher-collaborators: What if exploration of generalizations like these becomes the point of the lesson? How do elementary students understand claims for a class of numbers? How do they explain why such claims must

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be true? A few years into our inquiries, we realized our work was aligned with a broader movement to introduce algebraic thinking and proof into the elementary grades (Blanton et al., 2015; Cai & Knuth, 2011; Carpenter, et al., 2003; Carraher et al., 2006; Kaput et al., 2008). Along with other researchers, we understood a key aspect of early algebra to be generalized arithmetic, which we think of as the study of structure arising in arithmetic and quantitative reasoning. Not only does such study introduce basic principles of algebra, but investigating arithmetic structures helps young students come to see that an operation is more than a process or algorithm. It is a mathematical object in its own right (Kieran, 1989; Schifter, 2018; Sfard, 1991; Slavit, 1998). Furthermore, students' explanations about why such structures hold necessarily engage them in the question of how to prove a claim when you can't test all cases (Schifter et al., 2008; Stylianides, 2016; Stylianou et al., 2009).

BUILDING A MODEL FOR MATHEMATICAL ARGUMENT THROUGH CLASSROOM EPISODES

Data collected in our research collaborations with teachers included transcripts of videotaped classroom lessons. However, the bulk of the data we have collected across a series of projects funded by the National Science Foundation consisted of what we called "episode writing." Teachers recorded class and small-group dialogue and selected an episode within a lesson to transcribe. They then wrote a narrative based on the transcription, including discussion of their goals, the reasoning behind on-the-spot decisions, reflections on what transpired, and questions about student thinking. Project teachers wrote episodes about once every four weeks as a professional development exercise, at the same time providing us with a rich bank of data. Depending on the size of our teacher group, we collected between 60 and 250 episodes per year.

When we began episode writing in 1994, teachers wrote about whatever content they happened to be teaching at the time the assignment was due. As we became more focused in our research questions, teachers trained their episode writing on relevant topics. By 2001, groups of teachers in our projects regularly wrote about generalizations about the operations.

Through teachers' episode writing, we came to see the complexity of the task of having young students notice, express, and prove generalizations in the context of number and operations. For example, we sometimes read episodes in which students vigorously debated whether or why "it" worked. However, as readers, it wasn't clear to us what precisely "it" was, or if participants in the discussion shared the same idea of what they were debating. We suggested to teachers that it was important to have a clear statement of a conjecture before discussing whether it's true. Again, through episode writing, we began to see improved clarity when teachers took time to start a new investigation by allowing students to comment on whatever they noticed about a set of equations, expressions, or story problems specifically selected to embody a generalization (e.g. $5 + 7 = 12$, $5 + 8 = 13$, $5 + 9 = 14$). Having time to notice allowed more students to get on the same page before participating in conjecture formulation. More generally, we learned that constructing an argument for a generalization about an operation was not achieved in a single lesson, but needed days, with each lesson focused on a different aspect of the process. Eventually, we recognized the need to create a teaching model.

The research questions we posed for ourselves were: What phases can be incorporated into a teaching model that allows teachers and students to focus on key aspects of the process of constructing an argument? Are these phases consistent across different generalizations?

The first iteration of the teaching model was based on analysis of our existing corpus of classroom episodes, videotapes, and transcripts of class sessions of collaborating teachers. We coded these data sources to identify critical categories of the model and key facets of instruction within each category. Over the next two years, working with eight experienced teachers, we collected new classroom video and teacher-written episodes. We again coded episodes and transcripts to confirm and refine existing categories and to identify consistencies across classrooms that suggested new categories.

Once the categories and key aspects of instruction reached stability, we wrote lesson sequences based on the teaching model, each lesson identified with a phase of the model. A sequence consisted of about twenty 15- to 20-minute lessons taught outside of regular math class, much like the implementation of Number Talks (Parrish, 2010; Humphreys & Parker, 2015). In the next phase of the project, along with the more experienced teachers, we recruited additional participants new to the ideas of mathematical argument to implement the lesson sequences. Pre- and post-intervention written assessments of student learning were collected from all 18 classrooms. In addition, three students from each of the classrooms of the newly recruited teachers were selected for interviews at three points during the school year.

In this article, we illustrate our teaching model with classroom examples taken from videotaped lessons or teachers' written accounts based on audio recordings (episodes). We then cite some of the learning results from our projects in which teachers engaged their students in this content.

PHASES OF THE TEACHING MODEL

In order to help elementary grade students and their teachers focus in on key aspects of the process of formulating and proving conjectures about the operations, we created a teaching model of five phases (Russell et al., 2017):

1) ***Noticing regularities and patterns*** across multiple examples;
2) ***Articulating conjectures*** based on what students notice;
3) ***Representing examples*** with diagrams, pictures, physical models, and story contexts in order to understand the mathematical structure of their conjectures;
4) ***Constructing representation-based arguments*** for a class of numbers;
5) ***Comparing and contrasting operations*** by investigating analogous generalizations for another operation, recycling through phases 1 to 4.

We explicate each phase of the teaching model through scenes of a class working on a lesson sequence that addresses two generalizations: what happens to the sum when 1 is added to an addend, and what happens to the product when 1 is added to a factor. Although the students in these classes do not use algebraic notation, these generalizations can be expressed, for example, as

$$a + (b + 1) = (a + b) + 1$$
$$a(b + 1) = ab + a.$$

That is, students are investigating special cases of the associative property of addition and the distributive property of multiplication over addition.

In later years, students will come to understand the properties of the operations as axioms, statements taken to be true as the starting point for formal proofs. However, at this stage in their schooling, students are still learning about what the operations mean and the properties are still in question for them. As is demonstrated in the following scenes, when investigating such properties as associativity or distributivity, the question, "How do we know these statements are true?" is viable, productive, and engaging for elementary-aged students (Schifter et al, 2008).

Phase 1: Noticing regularities and patterns

The generalizations students study in these lessons are chosen to connect with core elementary curriculum about number and operations and are also informed by the patterns and regularities students in these grades notice on their own. For example, a first grader learning addition facts might say, "I know 5 + 5 = 10, so 5 + 6 = 11." In our lesson sequences, an idea such as this is not just a "trick" for solving certain problems but becomes an explicit focus of study. The lesson sequence begins by presenting students with a set of related equations illustrating the structure to be studied and asking them what they notice. In the first lesson, Ms. Cutler[1] presents this poster to her class.

7 + 5 = 12 7 + 6 = __	7 + 5 = 12 8 + 5 = __
9 + 4 = 13 9 + 5 = __	9 + 4 = 13 10 + 4 = __

What do you notice?

Note that the purpose of starting with these small numbers is not to review math facts students learned in earlier grades—the sums are easy for them to determine—but to work with familiar addition equations in order to explore a *structure of addition*. To launch the discussion, the class fills in the blanks and then begins to talk about what they see. Students

1 Ms. Cutler and her class are composites of teachers and students who enacted the lesson sequences. Phases 1 to 4 are illustrated with scenes from first, second, and third grade classes. Phase 5 is illustrated by scenes from third grade. (First and second graders investigated a related generalization for subtraction.) The data presented in this paper are drawn from students' work, teachers' written reports (episodes), and transcripts of videotaped class sessions.

mention that one number changes and one stays the same, and that the last number in the equation changes, too. The discussion goes on like this for a few minutes, students describing what they see, but not yet thinking about the *changes in relation to each other*, until one child, Evan, says, “Since 9 + 4 is 13, 9 + 5 *has* to be 1 more than 13.”

After further discussion, Pamela comments, “I was just wondering. How did Evan come up with the idea he had? Because these are not just everyday ideas that you come up with every day.”

Evan responds, “I’m not really sure. I just know it. It kind of seems obvious to me, so I didn’t think to think about it before.”

For Pamela, noticing patterns in the number system is a new and important kind of mathematical activity, while for Evan, some generalizations are so obvious as to be invisible. Most classrooms are likely to include students like Pamela and Evan—those who are just learning to look for arithmetic structures and those who are challenged to explicitly describe structures they have, until now, taken for granted. Whether individual students are for the first time finding patterns in the number system, searching for language to describe the patterns they have noticed, or beginning to articulate the mathematical relationships that underlie the pattern, all students in the class are engaged in the same discussion. The entire class benefits from the range of questions and ideas, although each student may be learning something different.

Phase 2: Articulating conjectures

Once students have had an opportunity to notice a pattern related to the behavior of an operation, they work on writing a clear conjecture about what that pattern is—a statement that would be comprehensible to someone outside the class who had not participated in the discussion. First, students work individually or in small groups to find language to describe their ideas. The teacher then selects several statements that contain elements the class may want to draw on as they work together to create a “class conjecture.” Ms. Cutler chooses the following three statements for the class to work with:

- In the first column, if the number goes up, the answer goes up.
- The first number goes up by 1 and the second number stays the same, so the last number goes up by 1.
- One number grows by 1 and the sum grows by 1.

This collection of statements includes the following important elements: two numbers increase by 1 (statements 2 and 3), one number stays the same (statement 2), one of the numbers that increases is the sum (statement 3), and there is a causal relationship between the changes (statements 1 and 2).

Once Ms. Cutler posts these three statements, the class discusses what parts are clear and what is confusing. The objective is not to provide students with the most precise and concise statement of a special case of the associative property. Rather, the goal is that students learn the language of generalization—what elements should be included to fully capture the idea, what wording could lead to a misinterpretation or overinterpretation, what information is too specific, too general, or not necessary. These initial statements are one aspect of what Cusi et al. (2011) call “algebraic babbling.” Students learn to communicate algebraic ideas by starting with everyday language and through collective discussion, verbalization, and argumentation, gradually learn to make their statements more precise and also become more proficient in the use of technical vocabulary and syntax.

Throughout the discussion, Ms. Cutler poses questions, challenging students to clarify referents to the word “number” and to become explicit about the operation acting on those numbers. Note that the operation is completely absent from the first two statements initially presented to the class and only implied by the word “sum” in the third. Ms. Cutler also suggests the term “addend” might be helpful to them. Students work to understand each other’s articulations by asking each other clarifying questions, paraphrasing classmates’ ideas, and suggesting alternative wording. At times in the discussion, students point to a list of numerical examples to explicate referents and check their conjecture against those examples. By the end of the lesson, the class agrees on the following statement as their conjecture:

In addition, if you increase one of the addends by 1 and keep the other addend the same, then the sum will also increase by 1.

Phase 3: Representing examples

In this phase, students use a variety of representations to explore specific cases in order to better understand the structure they have described and to lay the foundation for developing a general argument. Although students, both individually and collectively, have articulated and examined statements of their generalization, we can't assume that students know why this generalization will work for all numbers. They have seen that it is true in a few examples, and they may believe, based on their experience with addition, that it is true more generally, but they haven't yet dug into the structural connection between an increase in an addend and an increase in the sum. Some children may have constructed mental images of addition that convince them that their conjecture is necessarily true, but others may still be thinking of the two changes as "side by side" rather than causal. That is, they might recognize that the addend changes and the sum changes but not yet see that if the addend changes, then the sum must change in a comparable way. Most students in the elementary grades have never been asked to think about why a general idea about the structure of an operation is true and may have no experience with the connection between the two clauses of an "if-then" statement.

In the elementary grades, representations in the form of physical models, drawings, diagrams, and story contexts are tools for reasoning and communicating about structures in the number system. In the language of Luis Radford, "The awareness of these structures and their coordination entail a complex relationship between speech, forms of visualization and imagination, gesture, and activity on signs (e.g., number and proto-algebraic notations)" (2011, p. 23). Warren & Cooper hypothesize that "abstraction is facilitated by comparing different representations of the same mental model to identify commonalities that encompass the kernel of the mental model" (2009, p. 90). Moss & London McNab theorize that "the merging of the numerical and the visual provides the students with a new set of powerful insights that can underpin not only the early learning of a new mathematical domain but subsequent learning as well" (2011, p. 280). Representations that embody the relationships defined by the operations allow students to examine why the symbol patterns they identified work. They help students develop their own internal logic and connect the words of their conjecture to images of the operations and to its symbolic representations.

In this third phase of the model, students construct representations of numerical examples of their conjecture, and the teacher selects a sample of those representations which the class analyzes together. For example, Ms. Cutler gives to her class the following story context to illustrate.

> Ms. Sanchez has 12 bottle caps. Ms. Green has 8 bottle caps. When they put their collections together, they have 20.
>
> Ms. Sanchez found another bottle cap. Ms. Green still has 8 bottle caps but Ms. Sanchez now has 13 bottle caps. How many do they have together?

The next day begins with a discussion of the following student representation, which Ms. Cutler projects on the board (*Figure 1*). She first keeps the equations covered in order to focus the class on the drawing.

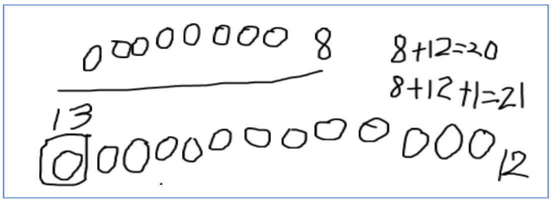

Figure 1. One student's representation of two related story problems

Ms. Cutler: What do you notice from this scholar's representation? And what does it tell you?

Jason: I see 12 circles but I see that 13 has a square so that means she added 1 more.

Ms. Cutler: Oh, interesting. Maddie, you're saying you agree. Can you say more?

Maddie: Like she has your 8 circles on the top and then she has 12 circles but the 1 that's the thirteenth one has a square.

Andre: It has a square because of … so we know that that's what she added.

Ms. Cutler: The problem tells us that Ms. Sanchez found another bottle cap and now she has 13. So instead of showing a whole other representation, drawing something completely different, what did you do, Latesha?

Latesha: I drew a box.

Ms. Cutler: Why?

Latesha: So people can know that one's the one that Ms. Sanchez found.

Ms. Cutler: Ah, to differentiate, to show you that this is the additional bottle cap that she found, and this will now be her 13th bottle cap. … Now (uncovering the equations) what do these equations tell us?

Mannie: It's like our conjecture because first it's 8 plus 12 equals 20. Now it's 8 plus 12 plus 1 equals 21. And we're adding 1 more, and our total is getting bigger.

Ms. Cutler: Can someone say more? … How is it like our conjecture? Maria?

Maria: Because our conjecture says that if you add one number to an addend, the number's gonna increase by … by the number you add. I mean, the sum's gonna increase by the number you add.

These young students are making sense of a representation that shows what happens to the sum as 8 + 12 changes to 8 + 13, and they are also beginning to see these related expressions as an example of a generalization about the operation of addition. The representation shows a specific pair of problems, but it also identifies key elements of the class's conjecture. As students discuss the representation, they begin to use more general language. Mannie says, "We're adding 1 more, and our total is getting bigger," no longer specifying the particular numbers. Maria appears to be extending the conjecture to adding amounts other than 1 to an addend, and she also expresses her idea in general terms, "lifting off" from the specific problem: "the sum's gonna increase by the number you add."

In a subsequent lesson, after reviewing her students' representations of 13+11 = 24 and 13+12 = 25, Ms. Cutler chooses three representations to bring to the class: a place-value representation (*Figure 2*), a drawing of cubes (*Figure 3*), and a number line (*Figure 4*). By examining different representations of the same structure, students begin to determine which elements of the representation are key to the structure and which are extraneous, in this way abstracting the structure from the specifics of the representations.

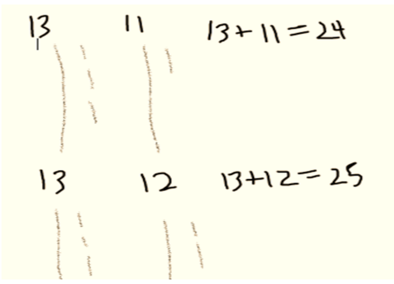

Figure 2. A place-value representation of two related equations

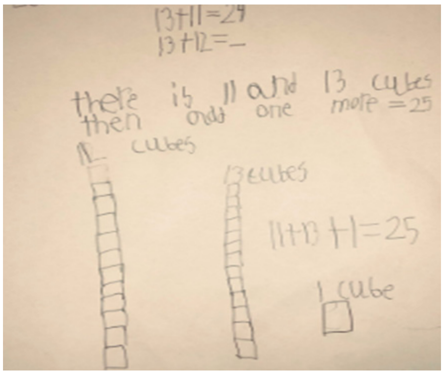

Figure 3. A drawing of cubes to represent two related equations

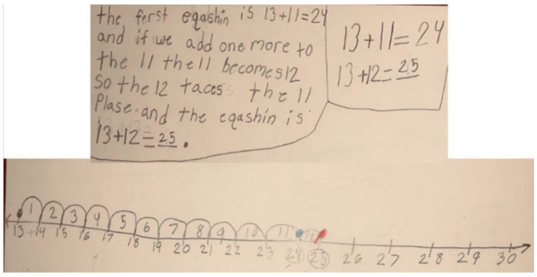

Figure 4. A number-line representation of two related equations

The class looks at each representation, one at a time, and responds to the following questions:

- Where do you see 13 and 11?
- How does the representation show 13 and 11 are added?
- Where do you see the sum?
- Where do you see the 1 that was added to 11?
- How does the representation show the sum increased by 1?
- How does the representation show our conjecture?

Designed to connect the representations to elements of the claim, these questions—which we have termed "core questions" in the lesson sequences—emphasize how the operation is shown. That is, they shift attention from an exclusive focus on quantities to include the action of the operation and the relationships it defines. Although these core questions may seem straightforward, we have found they take students into deeper understanding of the representation and often uncover confusions the class needs to work through about the mathematical relationships they are attempting to represent. Addressing the same questions for different representations, students learn to look for correspondences across representations, making explicit which aspects of the representations reveal the mathematical structure and which are incidental. By seeing the same idea represented in different forms, students develop a deeper understanding of the mathematical abstractions at the heart of their conjecture.

Students' investigations through representations often lead them to reconsider their conjecture. For example, students may start to notice that their conjecture is a special case of a more general one, as Maria suggested above. In another lesson, Bruce formulates a new conjecture: "If you add 2 to one of the addends, the answer, the sum has to be 2 more." Lan asserts, "it's not only 2, it's any number" and gives several examples: "you could even add 1000." The teacher then challenges the class to talk in pairs about these ideas. When the class comes back together, Aisha offers a new conjecture, "If you add any number to an addend, then your sum will be as much as the number you added more." The class has now returned to phase 2, as they work with this new conjecture and think about how to make it clearer and more precise. This illustrates how the phases of the teaching model interact. They are not a rigid sequence; work in one phase may stimulate work in another.

Phase 4: Constructing representation-based arguments

To prove a generalization within a number system, a mathematician states a premise and argues to a conclusion (the generalization), showing how each step from premise to conclusion is justified by a definition, fact, or already established principle, relying on the laws of arithmetic and algebraic notation to communicate the argument. Clearly, the tools of the mathematician's proof are not generally available to children in the elementary grades. At this level, most students are still coming to understand the kinds of actions that are modeled by the four basic operations. The laws of arithmetic cannot be the basis of their justifications when these laws are often still in question for them. Neither is algebraic notation typically available as a means for expressing generality.

However, young children are capable of justifying claims of generality. Reasoning from representations to justify their conjectures is an effective route for establishing general claims in the elementary classroom. Through studying examples of such justifications, we have identified three criteria that characterize representation-based arguments for generalizations about the operations (Russell et al., 2011; Schifter, 2009):

1. The meaning of the operation(s) involved is represented in diagrams, manipulatives, or story contexts.
2. The representation can accommodate a class of instances (for example, all whole numbers).
3. The conclusion of the claim follows from the structure of the representation.

In Phase 4 of the teaching model, having explored a variety of representations to illustrate specific examples of the conjecture, students then turn to these same representations in order to make an argument for the general claim. Rather than naming specific quantities, students describe the elements of their representations in such a way as to have them stand for a class of instances. For example, they might represent any number by drawing a container that holds an unspecified quantity or by imagining a stack of cubes to be any number, what Mason (1996) calls "seeing the general through the particular." Ms. Cutler's student, Otto, creates two stacks of cubes, each stack of a different color, represented here by gray and white (see *Figure 5*). He then duplicates the same two stacks and adds a single black cube to the second white stack.

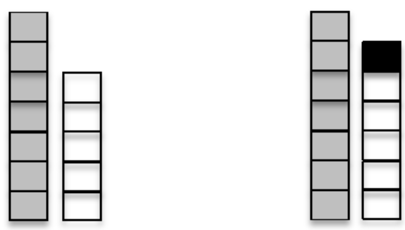

Figure 5. A representation-based argument for the generalization

Otto says his gray and white stacks can hold any number of cubes and explains, "This number [points to the first group of gray cubes] plus this number [the first group of white cubes] equals the sum. Then . . . the same number here [points to the second group of gray cubes] plus the same number plus one [points to the white cubes plus the black cube] equals the sum plus one."

Otto's argument satisfies the criteria of representation-based arguments: 1) The action of the operation of addition is represented as the joining of stacks of cubes. 2) The argument is independent of the number of gray or white cubes in his stacks. The representation is not just about specific quantities but can be used to show what happens with any whole numbers. Otto makes this clear by referring to each stack as "this number" rather than naming the actual number of cubes in the stacks; that is, the gray and white stacks can represent any two whole number addends. 3) The representation shows why the conjecture must be true: When the black cube is added to the white stack (1 is added to an addend), in that same action the combined gray and white stacks increase by one cube (the sum increases by 1).

In Phase 4, the class works through several such arguments, exploring how the same structure can be embodied in different forms of representation. For example, Cynthia builds her argument from a story problem the class worked on in Phase 3. "Augie and Kenisha were collecting shells. Augie had some amount and Kenisha had some amount and they put all the shells in a bag. On the way home, Augie found another shell. They brought home one extra shell."

Again, for each representation, Ms. Cutler takes students through the core questions: Where do you see the sum of two addends? Where is 1 added to one addend? Where is 1 added to the sum? How does this explain our conjecture? What may feel to adults like repetition is not to young students. They are building a network of connections from specific examples to generalizations, from quantities and actions with cubes, pictures, diagrams, and story contexts to abstract structures. At the same time, students are learning to address the question at the heart of mathematics: How do we know this is true?

Phase 5: Comparing and contrasting operations

If students have not had experience thinking about operations as objects that have their own properties and behaviors, they frequently think of generalizations as about *numbers* rather than about *operations*. That is, when they notice a generalization, they assume the same number patterns will occur, whether they insert the symbol +, –, ×, or ÷. Such confusion about the operations can result in consistent procedural errors. Indeed, common errors in subtraction or multiplication can be interpreted as an application of structures that apply only to addition. Elementary teachers report such common misunderstandings as the following.

Since 145 – 100 = 45, 145 – 98 must be 2 less than 45.
Since 10 × 7 = 70, the product of 9 × 7 must be 1 less than 70.
Since 52 + 41 = (50 + 40) + (2 + 1), 52 × 41 must equal (50 × 40) + (2 × 1).

Similar misunderstandings carry into students' later study of algebra. Any algebra teacher will recognize such common errors as:

$$a - (b - c) = (a - b) - c$$
$$a \times (b - 1) = ab - 1$$
$$(a + b) \times (c + d) = ac + bd$$

For this reason, it is important to explore sets of related generalizations that highlight the differences between operations and the limits of the conjecture students have just proved. As they work through the lesson sequences with their class, our collaborating teachers find that once students have investigated a generalization about one operation, they are often surprised to discover that the same pattern does not hold for another operation. For example, they've learned that increasing an addend increases the sum by the same amount, so they expect the same thing to occur in subtraction or multiplication. As they try to apply the same pattern to another operation, they run into counterexamples and realize they must look for another pattern and develop a different conjecture and mathematical argument. Ms. Cutler decides to have her class investigate what happens to the product when 1 is added to a factor because multiplication is a focus in her regular third-grade math curriculum.

Ms. Cutler begins this second investigation by presenting two pairs of equations and asks the class what they notice.

7 × 3 = 21	7 × 3 = 21
8 × 3 = 24	7 × 4 = 28
What do you notice?	

After some discussion, she asks students to work individually in response to the prompt: *In a multiplication problem, if you add 1 to a factor, what will happen to the product*?

The next day, Ms. Cutler presents a set of students' statements, one by one, asking the author of the statement to read it aloud and inviting classmates' comments.

Karsyn: *When I add 1 to the factor, the product increases by 3.*

The class recognizes that this is a true description of one pair of the specific equations given to them, but by this point in their work they know they are looking for a general claim. For example, if 7 × 4 is changed to 8 × 4, the product does not increase by 3.

Janie: *When I add 1 to the first factor, the product gets bigger.*

Janie had written a true statement, but the class suggests they can specify how much the product increases.

Jaxon: *When I add 1 to the first factor, the product increases by how much we're multiplying by.*

Layla asks, "What do you mean by, how much we're multiplying by?"

Jaxon: I said that because 7 × 3, you're multiplying 3 seven times. Since 7 increases by 1, the product goes up by 1 group of 3. [Jaxon goes to the board and repeats his statement pointing to numbers in the equations.]

Ms. Cutler posts the next statement.

Robert: *When I add 1 to the first factor, the product increases by the second.*

Jaxon wants to know what Robert means by "the second": the second what? Before Robert can answer, Ms. Cutler reveals Arya's statement.

Arya: *When I add 1 to the first factor, the product increases by one of the second factor.*

"Wow," several students in the class call out.

Ms. Cutler: Why is everyone so surprised right now?

Janie: They wrote the same thing.

Dexter: Similar.

Zoe: Robert wrote, "the second," but Arya wrote what it meant. She wrote what "the second" meant.

Caleb: What does she mean by "one of the second factor"?

Arya: Because we're increasing the first factor by 1, the product increases by 1 of the second factor, because we're adding one more group to the product.

Ms. Cutler: I think everyone was blown away by Robert and Arya, not only because they wrote similar things, but because Arya's statement answered the question Jaxon asked. I want you to think about how language is important. One word helped us understand. You can ask, Am I being as precise as possible without being too specific?

Through their exploration of specific examples and a statement of the general claim, many students are beginning to understand how increasing a factor is related to the number of groups implied by multiplication. As they continue to explore different pairs of equations, including those that show a change in the second factor, the class lands on the claim, *When I add 1 to a factor, the product increases by one of the other factor.*

Their next task is to create story situations and diagrams to illustrate the claim. After time to work alone or in pairs, the class analyzes different images. For example, to represent 7 × 5, Zoe draws a picture like this for her story about 7 jewelry boxes that each contain 5 jewels (*Figure 6*).

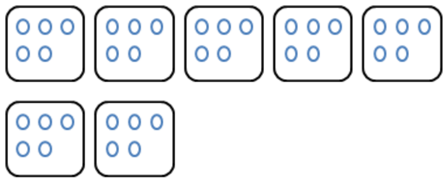

Figure 6. Zoe's representation of 7 × 5

To change the story to 8 × 5, she adds another box, increasing the product by 5 (*Figure 7*). She shows the new box of jewels by coloring it red (shown below as gray).

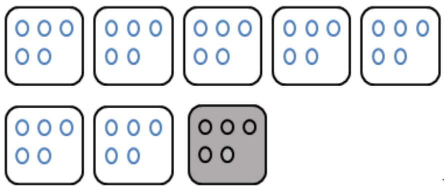

Figure 7. Zoe's representation that shows 7 × 5 and 8 × 5

Zoe goes back to the original story for 7 × 5, and then, to change it to 7 × 6, she explains that 1 more jewel was added to each box (*Figure 8*). She indicates the additional jewel in her picture by showing a shaded-in dot in each group. When 5 increased to 6, there are 7 additional jewels.

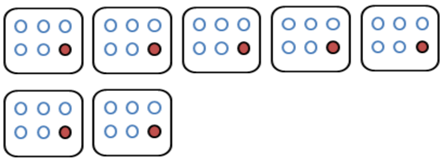

Figure 8. Zoe's representation that shows 7 × 5 and 7 × 6

Kenesha makes up a story about chairs arranged in rows. Her diagram of the three situations—7 rows with 5 in each row (7 × 5), 8 rows with 5 in each row (8 × 5), and 7 rows with 6 in each row (7 × 6)—looks like this (*Figure 9*).

XXXXX	XXXXX	XXXXX**X**
XXXXX	XXXXX	XXXXX**X**
XXXXX	XXXXX	XXXXX**X**
XXXXX	XXXXX	XXXXX**X**
XXXXX	XXXXX	XXXXX**X**
XXXXX	XXXXX	XXXXX**X**
XXXXX	XXXXX	XXXXX**X**
	XXXXX	

Figure 9. Kenesha's representation of 7 × 5, 8 × 5, and 7 × 6

The class examines the variety of representations by answering the same set of core questions about each representation:

- Where do you see 7 and 5?
- How does the representation show 7 and 5 are multiplied?
- Where do you see the product?
- Where do you see the 1 that was added to 7?
- How does the representation show what was added to the product?
- Where do you see the 1 that was added to 5?
- How does the representation show what was added to the product?
- How does the representation illustrate our conjecture?

As students describe and compare representations and relate them to their conjecture, the language in students' arguments starts to become general: "When 1 was added to 7, there's one more row, so the number of chairs increases by the number in each row. When 1 was added to 5, there's one more chair in each row, so the number of chairs increases by the number of rows." Eventually, they create an argument to justify the conjecture: Adding 1 to the first factor is like adding a jewelry box or adding a row of chairs. Adding 1 to the second factor is like adding 1 jewel to each box or 1 chair to each row.

At the conclusion of the exploration of multiplication, Ms. Cutler takes the class back to their exploration of addition and reminds them of their initial conjecture about adding 1 to an addend. "We were talking about the addends changing by 1 and what happens to the sum. Now we're talking about the factors changing by 1 and what happens to the product. How different are those situations?"

Mazie suggests, "When you do multiplication, you have to think about groups, and that's different than when we are doing sums." Several students express variants of this idea with their own words. By asking the class to think about addition and multiplication together, Ms. Cutler is making sure her students are thinking about the operations as distinct entities with different behaviors. Operations are not simply instructions to calculate according to a certain set of steps. By developing representation-based arguments about each operation, students are learning to consider the operations as objects of investigation in their own right.

ASSESSMENT RESULTS

One common objection to working on the ideas of proof in the elementary grades is that the curriculum is already very full—why not leave such an advanced and complex practice to later grades, after students have mastered basic computation? Our research demonstrates that working on representation-based argument, with a focus on the behavior of the operations, not only introduces students to core mathematical practices, but also enhances the study of content at the heart of the elementary curriculum and develops a foundation for connections to later study of algebra.

Based on what our research team and collaborating teachers were learning through implementing our model of mathematical argument in the classroom, we designed a professional development course (Russell et al, 2012) with the following goals: to help teachers understand and look for structural properties implicit in students' work in number and operations, to bring students' attention to such properties, and to support students to articulate, represent, and create mathematical explanations of the properties. We offered the course on-line as part of the research grant under which we were developing these ideas.

Pre- and post-course assessment data were collected from 601 students of 36 participating teachers of grades K to 5 and, as comparison, from 249 students of 16 non-participating teachers in the same school systems (Russell et al., 2011; Russell et al., in preparation). On some items, students were asked to choose two equivalent expressions out of three given expressions—for example, 6 × 7, 3 × 14, 5 × 8—and explain their choice. Other problems were presented in a story context: "Marissa says, 'Problems like 27 + 9 – 9 = 27 are easy because when you add a number and then take that same number away you don't really have to do anything.' Do you think her idea always works? Explain." The numbers and operations in the problems were chosen appropriately for each grade level. Student responses were coded for the type of explanation they provided: a) no explanation; b) a computational explanation; or c) a relational explanation, that is, an explanation that refers to mathematical structure. For instance, to explain why 6 × 7 and 3 × 14 are equal, a student could carry out both computations, showing that each expression equals 42, or the student could give a relational explanation, e.g.,"If there is a doubled factor and a halved factor, the two are equal." At post-test (see *Table 1*), grades 3-5 students of teachers in the Participant Group provided significantly more relational explanations than in the Comparison Group on most of these problems. While grade 2 students in the Participant Group also provided more relational explanations than students in the Comparison Group on all problems, the comparison reached statistical significance for only one of the problems.

The first two items in *Table 1* required students to identify two of three expressions that were equivalent. The other three items were embedded in story contexts.

These results are promising, especially in light of significant challenges to assessing students in this study: the Participant teachers were taking an online course, so had only long-distance interactions with the facilitators and with each other; the assessments were done during the same year the teachers were taking the course, that is, the first time they had implemented our teaching model for mathematical argument in their classrooms; and we were limited to what we could learn from a paper and pencil assessment without any opportunity to question students about their responses.

As part of a subsequent teaching experiment with twelve teachers of grades 2 to 5, teachers implemented an instructional sequence to explore two contrasting generalizations such as those illustrated in the case of Ms. Cutler. Individual interviews were conducted with three students from each classroom (n=36), representing the range of learners, characterized in terms of strong, average, or weak in grade-level computation. In one strand of the interview, students were given pairs of subtraction problems (for example, 10–3= 7; 10 – 4 = ?) that illustrate a structural property not explored in the instructional sequences their class studied: Given a subtraction expression, if the second term (the subtrahend) increases by 1, the difference decreases by 1. Students were asked to describe what they noticed, come up with other pairs of problems that illustrate the same behavior,

Table 1. Proportion of Students by Grade and Group Providing Relational Explanations at Posttest

Item	Comparison 3-5 n = 109	Participant Gr 3-5 n = 473	Comparison Gr 2 n = 41	Participant Gr 2 n = 101
9-5/10-6	0.05	0.26*	0.00	0.04
6×7/3×14	0.13	0.29*	NA	NA
27+9-9=27	0.36	0.68*	0.44	0.50
35+12=25+___	0.11	0.31*	0.02	0.18*
46+17=36+___	0.50	0.55	0.27	0.36

*(Grade 2 - $X^2_{1;.05}$ = 5.9) (Grades 3 -5 $X^2_{1;.05}$ = 37.75; 57.50; 19.6; 28.7)

state a conjecture, and use a representation to explain why the conjecture must be true.

In the analysis of interview data (Higgins et al., in preparation), one of the dimensions that distinguished students' conjectures was salience of the operation, defined as the degree to which students attend to the behavior of the operation as opposed to focusing exclusively on the numbers when making generalizations and articulating conjectures. Some students articulated generalizations that were fundamentally about the operation: "When you have the same numbers, once you subtract more, you'll have less. And if you subtract less, you'll have more." Other students showed no evidence of attention to the operation: "The numbers in the middle, you just add 1. Then the answer you take away 1. The first numbers are the same." This student reads the equations, 10 – 3 = 7 and 10 – 4 = 6, from left to right, identifying the two 10s as the "first numbers," 3 and 4 as the "numbers in the middle," and 7 and 6 as "the answer." She gives no indication that the changes in the subtrahend and the difference have anything to do with each other or that the relationship between the two changes is a consequence of the behavior of subtraction.

In interviews conducted at the beginning of the year, lack of salience of the operation was found for close to half the students. After working on lesson sequences that explored generalizations about the operations (but not the one about subtraction used in the interview), more students explicitly referenced the operation or talked about what they were noticing in operation-specific terms. The operation was no longer just part of the background, but became something that students realized they needed to attend to when articulating what they were noticing.

Data from both studies suggest that lessons in which students investigate the distinct structures of each operation help to make the operations a salient object in students' mathematical experience. The written assessment data reveal that, once students have an opportunity to explore and represent properties of the operations, they rely on characteristics of the operation to explain the equivalence of arithmetic expressions. The interview data demonstrate how, when students notice patterns across calculation problems, they recognize the pattern as related to the structure of a given operation.

The results are positive. Nevertheless, we must note that in the post-intervention written assessments, there were students who continued to rely on computation to prove the equivalence of two expressions. In the interview data, even after the intervention, there were still students at each grade level who produced conjectures in which the operation was invisible. We need to continue to learn about the extent and characteristics of instruction focused on mathematical argument that make the operations salient objects in all students' mathematical experience.

CONCLUSION

As the examples and data presented in this paper illustrate, the task of proving generalizations about numbers and operations engages students in many facets of mathematical proficiency (Kilpatrick, et al., 2001). The goal of having students articulate and justify claims about how the operations behave is not merely one more piece of content for teachers to squeeze into an already full agenda. Through these discussions, students develop richer understandings of the meaning of the operations and their relationships, which in turn support greater flexibility with computational procedures.

The challenge to develop mathematical arguments is not only an enrichment activity reserved for advanced students. As one teacher wrote: "When I began to work on generalizations with my students, I noticed a shift in my less capable learners." When generalizations are made explicit—through language and through visual representations used to justify them—they become available to more students and provide foundation for greater computational fluency. Furthermore, the habit of creating a representation when a mathematical question arises supports students in reasoning through their confusions. They come to see mathematics as sensible and develop confidence in their own efficacy.

The study of number and operations extends beyond efficient computation to the excitement of making and proving conjectures about mathematical relationships that apply to an infinite class of numbers. All students, including students who see themselves as "good at math" because they are quick and accurate at computation, can be challenged and stimulated by this content. As one teacher explained, "Students develop a habit of mind of looking beyond the activity to search for something more, some broader mathematical context to fit the experience into."

For the full range of students, such an approach to proof deepens students' understanding of the number system while engaging them in a process central to the practice mathematics. As they experience a variety of modes of coming to believe the truth of a claim, reasoning becomes the standard for its acceptance. They learn to privilege mathematical reasoning over appeals to authority or testing instances.

In the course of our projects focused on noticing, articulating, and proving generalizations about the operations, we began to notice how, for elementary students, the work is highly collaborative. In these lessons, students typically build on each other's thinking as they notice patterns, develop and revise a class conjecture, critique and revise representations, and use those representations to build arguments. We hear students explicitly referring to each other's statements and representations, describing emerging ideas as "our" conjecture or "our" argument. Spurred by work in the field on mathematical agency, and a relatively new and unexplored concept of "collective mathematical agency" (Aguirre et al., 2013), we began to focus on how to describe and analyze the collective nature of this work. In the past few years, we have been pursuing more intensive research in urban classrooms (from which several of the examples in this paper come), focusing on the intersection of mathematical argument and collective mathematical agency. In our current work, we are documenting how teachers in schools that have been labelled as "failing" support their students to engage in high-level mathematical reasoning. The work ahead for the mathematics education community is to bring educational systems to prioritize the engagement of all students in tasks that require reasoning and justification, and to help teachers learn to recognize the sense-making in their students' representations and explanations.

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