Journal of Educational Research in Mathematics
2020, Special Issue, 115~ 134

Teachers' Views about the Role of Examples in Proving-related Activities

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ABSTRACT

Examples play a critical role in mathematical practice, in general, and in proving-related activities (e.g., developing conjectures, exploring conjectures, justifying conjectures), in particular. Yet, despite the critical role examples play in proving-related activity, we contend that students typically receive very little, if any, explicit instruction on how to become more deliberate and strategic in their use of examples. The goal of the study reported here was to explore teachers' beliefs about the role examples play in proving-related activities, and the instructional practices they implement to foster the development of students' abilities to strategically think about and productively use examples. Fifty-four middle school mathematics teachers responded to a series of on-line survey questions that focused on the role and use of examples during proving-related classroom activities. We found that many teachers have limited views of what it means to use examples strategically during proving-related activities, and that they tended not to provide explicit instruction designed to help students learn to strategically think about and productively use examples during their engagement in proving-related activities. The findings suggest the need for both professional development and curricular resources to support teacher efforts to help their students learn to strategically think about and productively use examples during proving-related activities.

Key Words: Proving, Teacher beliefs, Example use

TEACHERS' VIEWS ABOUT THE ROLE OF EXAMPLES IN PROVING-RELATED ACTIVITIES

Proving is fundamental to mathematical practice and also plays an important role in mathematical learning. Accordingly, mathematics education scholars (e.g., Knuth, 2002a, 2002b; Stylianides, Bieda, & Morselli, 2016; Stylianides, Stylianides, & Weber, 2016; Zaslavsky, Nickerson, Stylianides, Kidron, & Winicki, 2012) and reform initiatives (e.g., Council of Chief State School Officers [CCSSO], 2010; Department of Education [DoE], 2014; Ministry of Education, 2015; National Council of Teachers of Mathematics, 2000) have increasingly called for proving-related activities (e.g., developing conjectures, exploring conjectures, proving conjectures) to play a more central role in the mathematics experiences of students at all grade levels. Reform initiatives in the United States, *Common Core State Standards for Mathematics*, in England, *Mathematics Programmes of Study*, and in Korea, *Korean Mathematics Curriculum*, for example, advocate similar recommendations with regard to proving-related activities in school mathematics:

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Please cite this article as: Knuth, E., Kim, H., Zaslavsky, O., Vinsonhaler, R., Gaddis, D., & Fernandez, L. Teachers' views about the role of examples in proving-related activities. *Journal of Educational Research in Mathematics* (2020). https://doi.org/10.29275/jerm.2020.08.sp.1.115

> One hallmark of mathematical understanding is the ability to justify. … Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. (CCSSO, pp. 4–6)

> The national curriculum for mathematics aims to ensure that all pupils … reason mathematically by following a line of enquiry, conjecturing relationships and generalisations, and developing an argument, justification or proof using mathematical language. (DoE, p. 3)

> Instruction and curriculum should help students conjecture mathematical facts on their own through using plausible reasoning such as induction and analogy in observation and inquiry situations, and justify them based on appropriate evidence. (Ministry of Education)

Yet, despite almost two decades of calls to elevate the status and role of proof in school mathematics, students continue to struggle learning to prove (e.g., Harel & Sowder, 2007; Knuth, Choppin, & Bieda, 2009a; Reid & Knipping, 2010; Stylianides, Stylianides, & Weber, 2016), and teachers as well struggle to facilitate the development of students' learning to prove (e.g., Bieda, 2010; Bieda, Drwencke, & Picard, 2014; Cirillo, 2011; Stylianides, Stylianides, & Shilling-Traina, 2013).

Researchers have suggested that students' overreliance on examples as a means of conviction and justification underlies students' difficulties in learning to prove (e.g., Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009b) and, not surprisingly, instructional approaches designed to help students understand the limitations of examples as well as the need for proof have been advocated (e.g., Sowder & Harel, 1998; Stylianides & Stylianides, 2009; Zaslavsky et al., 2012). Although such instructional approaches may help students learn the limitations of examples and, consequently, understand the need for proof, they also tend to treat examples-based reasoning as a stumbling block to quickly overcome rather than as an essential aspect of proving-related activity. In contrast, we view examples-based reasoning as playing a critical and necessary role in the development, exploration, and understanding of conjectures, as well as in the development of proofs of those conjectures. Moreover, we contend that curriculum and instruction must be explicitly designed to help students learn to strategically think about and productively use examples during their engagement in proving-related activities.

THE COMPLEX INTERPLAY OF INTERPLAY OF EXAMPLE USE AND PROVING-RELATED

Examples play a critical role in mathematical practice; indeed, the time spent thinking about and analyzing examples during proving-related activities can provide not only a deeper understanding of a conjecture, but also insight into the development of a proof (e.g., Lakatos, 1976; Polya, 1954). In fact, Epstein and Levy (1995) contend that "Most mathematicians spend a lot of time thinking about and analyzing particular examples. […] It is probably the case that most significant advances in mathematics have arisen from experimentation with examples" (p. 6). Mathematicians' example use during proving-related activities has been characterized as including traits or approaches such as a metacognitive awareness of the relationship between their example-use activities and proving-related activities (Lockwood, Ellis, & Lynch, 2016); a complex, non-linear engagement of example use during proving-related activities (Antonini, 2006; Weber, 2008); and a systematic, deliberate, and reflective approach to example use during proving-related activities (Lockwood, Ellis, & Knuth, 2013; Weber & Mejia-Ramos, 2011).

1. The Role of Examples in Proving-related Activities

Mathematics education scholars have noted a number of roles and uses of examples related to various aspects of proving (e.g., Alcock & Inglis, 2008; Buchbinder & Zaslavsky, 2018; Ellis, Lockwood, Williams, Dogan, & Knuth, 2012; Iannone, Inglis, Mejia-Ramos, Simpson, & Weber, 2011; Sandefur, Mason, Stylianides, & Watson, 2013). In a recent project, we extended prior research and developed a comprehensive analytic framework for characterizing the roles and uses of examples in the proving-related activities of middle and high school students, undergraduate students, and mathematicians (for an overview, see Knuth, Zaslavsky, & Ellis, 2019). The framework is comprised of five major categories: the intended purpose an example serves, the criteria used for choosing an example, the affordances that may result from using the example(s), the strategies employed during example use, and the transitions or shifts in proving-related activity as a result of example use. Each major category is also delineated into several sub-categories that further differentiate the activities within the major categories (for example, within the major category of *affordances of example use*, sub-categories included gaining insight, producing a generalization, producing a viable proof, and developing a new or revised conjecture). An in-depth description of the framework is beyond the scope of this paper (see Ellis et al., 2019, for a detailed description), however, the preceding outline of the framework with its various categories and sub-categories serves to highlight the complexity and breadth of example use in proving-related activities. The framework also provides a means to contrast various aspects of example use among learners as well as to identify patterns of example use that are associated with productive (and less productive) proving-related activities.

2. Productive (and Less Productive) Uses of Examples in Proving-related Activities

We characterize productive example use during proving-related activities as use that helps a learner make progress toward the development of a proof. In particular, we view productive example use as activity that leads to a deeper understanding of a conjecture; an insight with regard to the development of a proof; an awareness of a generalization or underlying structure; the generation of a counterexample; the development of a new or revised conjecture; or an appreciation for the need for proof. As might be expected, the extent that example use is productive in the proving-related activities of sophisticated learners (e.g., mathematicians) often stands in contrast to the extent that example use is productive in the proving-related activities of less sophisticated learners (e.g., students). In our recent work, we identified patterns of students' productive (and less productive) example use among both successful and unsuccessful provers (Aricha-Metzer & Zaslavsky, 2019; Ellis et al., 2019; Lynch & Lockwood, 2019; Ozgur, Ellis, Vinsonhaler, Dogan, & Knuth, 2019).

To illustrate productive example-use, the following two excerpts from our task-based interviews with secondary school students demonstrate example use activity characterized within the *affordances of example use category* and the sub-categories of *gaining insight* and *producing a generalization*. In the first case (*see Figure 1*), the student's use of an example initially served as a means to understand the conjecture as well as to test validity of the conjecture, he gained a key insight which then led him to view the example as a particular instance of the general case (in other words, the example served as a generic example; cf. Leron & Zaslavsky, 2013), thus enabling him to see why the conjecture must always be true.

<table>
<tr><td colspan="2">Task: Eric noticed that when he adds any whole number to the number that comes two before it and the number that comes two after it, the answer is always equal to three times the number he started with. Do you agree with Eric that this conjecture is always true? Why?</td></tr>
<tr><td>Student work:
Okay, so, like 7,
7
and 2 before 7 is 5,
7 + 5
and then 2 after it is 9.
7 + 5 + 9
So that equals 21, right?
7 + 5 + 9 = 21
And so it's 7 times 3 is 21, and the point is that if you subtract 2 and you add 2, they cancel each other out. Oh, so you can move 2 from here to here, and then, it would just be 7+7+7.
+2
7 + 5 + 9 = 21
7 × 3 = 21
So it will always work.</td><td>Commentary:
The student initially tested the truth of the conjecture using a particular example.

After writing the expression that represents the conjecture, he makes the initial insight that subtracting 2 and adding 2 to the starting number will cancel each other out. Although he does not use this insight to further his progress toward a proof, the insight could lead to such progress (e.g., (x - 2) + x + (x + 2) = 3x).

He then notices that he can take 2 from the last number, 9, and add it to the second number, 5, with the result being three 7s.

Once he makes the latter observation, he immediately concludes that the conjecture will always be true.</td></tr>
</table>

Figure 1. An example of productive example-use: key insight and a generic example.

In the second case (see *Figure 2*), the student tests the conjecture with two sets of different examples, finds that the conjecture is true for both sets of examples, and then gains the key insight that for every set of three consecutive numbers there will always be one number divisible by two and one number divisible by three. In both cases, neither student produced a formal proof of the presented conjectures, yet their example use did lead to key insights about the underlying mathematical structure of each conjecture, insights that are essential to understanding why the conjecture must be true and for making progress toward the development of a proof.

In contrast, the following two excerpts illustrate less productive use of examples. In the first case (see *Figure 3*), the student's example use led him to be convinced that the given conjecture is true based on checking five different sets of diverse examples, but his use of examples did not lead him to appreciate the need for a proof (as he seemed convinced by the examples themselves) nor did his use of examples help him make progress toward the development of a proof. In the second example (see *Figure 4*), the student forgoes the use of examples and immediately represents the conjecture symbolically, proceeds to simplify the symbolic expressions, and then reaches an impasse in which he is unable to make further progress toward the development of a proof. In this case, the students lack of example use may have limited the student from gaining the key insight that was observed in *Figure 2*.

The preceding excerpts illustrate example use that enabled students to move productively (or not) toward the development of a proof (or at least to key insights toward that development). Throughout our project's data corpus, instances of students' productive use of examples provided evidence that some students are able to use examples productively,

Task: Trevor came up with a conjecture that states: If you multiply any three consecutive numbers together, the answer will be a multiple of 6. Do you think Trevor's conjecture is true? Why?	
Student work: The student finds the product of 2 • 3 • 4, and notes that the product is divisible by 6. And then finds the product of 10 • 11 • 12, and notes that the product is also divisible by 6. He then goes on to say "It worked for 2, 3, and 4, and for 10, 11, and 12. For each one there was a number divisible by 2 and a number divisible by 3, which means final answer will be divisible by 6."	Commentary: The student initially tested the truth of the conjecture using two sets of examples, and found that in each case the conjecture was true. He then noticed that for each set of three consecutive numbers, there will be one number divisible by 2 and one number divisible by 3, and concluded that the product then has to be divisible by 6.

Figure 2. An example of productive example-use: key insight

Task: Tyson came up with a conjecture that states: If you add any number of consecutive whole numbers together, the sum will be divisible by however many numbers you added up. Do you think the conjecture will be true for any 5 consecutive numbers? Why?	
Student work: [The student tried the following five examples and, although not shown below, proceeded to check that each sum was divisible by 5.] 10 + 11 + 12 + 13 + 14 (-1) + (-2) + (-3) + (-4) + (-5) 347 + 348 + 349 + 350 + 351 300 + 301 + 302 + 303 + 304 102,573 + 102,574 + 102,575 + 102,576 + 102,577 Interviewer: Do you think the conjecture will be true for any five consecutive numbers? Student: Probably yeah. Yeah. Interviewer: And you believe it is true because? Student: Because I did a bunch of trials that go really far into the depths of numbers, including negatives, which kind of sealed the deal for me because negatives are really different from positives.	Commentary: The student's focus seemed to be centered on confirming that the conjecture for five consecutive integers was true. His use of examples was enough to convince him that the conjecture was true because he tested the conjecture with a diverse set of examples (e.g., large numbers, negative numbers). And as a result, there was no need to test further examples or to justify beyond examples. The student did not seem to think about or analyze the examples in a way that might have allowed him to see, for example, structural characteristics of the examples that may have led to an insight regarding the development of a proof (e.g., that the first and last numbers are ±2 from the middle number, and the second and penultimate number are ±1 from the middle number). In this case, the example use was less productive as it did not help the student progress toward the development of a proof.

Figure 3. An example of less productive example-use.

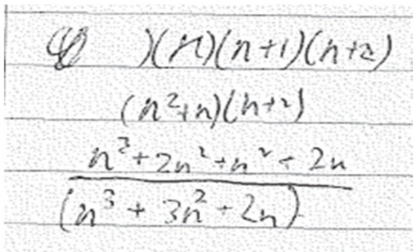
Task: Trevor came up with a conjecture that states: If you multiply any three consecutive numbers together, the answer will be a multiple of 6. Do you think Trevor's conjecture is true? Why?	
Student work: 	Commentary: The student represented the conjecture algebraically, expanded his initial algebraic expression, and then reached an impasse in which he was unable to make sense of the representation in relation to the claim. At this point, the student was unable to make any further progress toward a proof.

Figure 4. A second example of less productive example-use.

while the (more common) instances of students' less or unproductive use of examples suggest the need to explicitly develop both students' awareness of productive example use and their abilities to use examples productively. The latter suggestions are underscored in results from a study by Sandefur and colleagues (2013) who found that a characteristic exhibited by students for whom example use was productive was *sufficient experience* regarding the utility of example use in proving. In fact, we posit that students' lack of instructional guidance and experience with using examples productively during proving-related activities is the overarching reason that many students fail to learn from examples during proving-related activities.

3. Fostering the Development of Productive Example Use

To date, the majority of research related to the use of examples in proving-related activities has focused primarily on identifying ways in which mathematicians and students use examples, and very little research has focused on the nature of curriculum and instruction that may facilitate the development of students' abilities to strategically think about and productively use examples as they engage in proving-related activities. And ultimately, without such research and the correspondingly informed instructional practices and curricular materials, most students are unlikely to develop the necessary abilities to productively use examples in proving-related activities. Indeed, the contrast between the role examples play in the work of mathematicians and in the work of students is not surprising given that students typically receive very little, if any, explicit instruction on how to become more deliberate and strategic in their use of examples to support their efforts in learning to prove. Although research, including our own, has documented productive example use in the proving-related activities of a minority of students, an open and critical question is whether such productive example use is teachable and learnable. An important first step toward addressing this question is to understand teachers' views about the role examples play in proving-related activities—the goal of the work presented in this paper.

RESEARCH GOAL & QUESTIONS

The specific goal of the research reported here is to understand middle school teachers' thinking about and instructional practices related to example-use in proving-related activities.

In particular, the research sought to address the following questions:

1. What is the nature of middle school teachers' thinking about and instructional practices related to example-use in proving-related activities?
2. To what extent do middle school teachers, through instructional guidance, foster students' strategic thinking about and productive use of examples during proving-related activities?

METHOD

1. Participants

The participants were mathematics teachers (n=54) from middle schools (students aged 11-14 years) located in the Mid-South region of the United States. Twenty-five teachers taught Grade 6, twenty-two teachers taught Grade 7, and twenty teachers taught Grade 8 (note that some teachers taught multiple grades). The sample included thirteen beginning teachers (1-3 years of teaching experience), twenty-one mid-career teachers (4-10 years of experience), and twenty veteran teachers (10+ years of experience). The middle school curricula used by the teachers varied, however, the curricula were all aligned with the State's curriculum framework for middle school mathematics, which includes explicit attention to proving-related activities (for example, the framework notes that middle school students are expected to "display, explain, or justify mathematical ideas and arguments").

Although proving-related activities are expected to play a more prominent and consistent role throughout K-12 mathematics education, the rationale for focusing on middle school, in particular, rather than K-12, in general, is due to middle school being a particularly important time period as it represents the transition from the concrete arithmetic thinking of elementary school to more advanced abstract mathematical thinking of high school. Moreover, middle school curricula often provide ample opportunities to engage students in example use during proving-related activities (e.g., Bieda, 2010; Stylianides, 2009), and middle school students often rely on examples as a primary means of conviction and justification (e.g., Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009a; Knuth, Kalish, Ellis, Williams, & Felton, 2011).

2. Data Collection

Teachers responded to an on-line survey with a series of questions focused on the role and use of examples during proving-related classroom activities (see the Appendix for the full set of survey questions). The survey questions included bothforced choice responses (often with Likert scale type choices) and open-ended responses. In the former case, for example, teachers were asked the extent to which they thought students are strategic in thinking about their use of examples when developing, exploring, or justifying conjectures, and given three Likert scale choices—not very strategic, somewhat strategic, and very strategic. In the latter case, for example, teachers were asked what purpose they thought students' example use serves.

The purpose of the on-line survey was to learn about teachers' self-reported instructional practices with respect to example use during proving-related activities (both what they might say and what they might do) as well as their thoughts about their students' example use during their engagement in proving-related activities. In addition, we were also interested in examining teachers' thoughts about specific example use approaches that might be used for the exploration and justification of presented conjectures, approaches ranging from empirically-based justifications to generic example-based justifications.

3. Data Analyses

The on-line survey analysis drew on our recent project (example-use analytic framework, productive/less productive example use) as well as the research literature related to proving and example use to establish a set of a priori coding categories. For example, teachers' responses about the roles and purposes of students' example use were initially coded using the primary categories and their associated sub-categories from the aforementioned example-use analytic framework (e.g., purpose of example use to *test the truth* of a conjecture; purpose of example use to *understand* a conjecture). The analysis focused primarily on teachers' self-reported views about the role examples play in proving-related activities, about students' thinking about and use of examples, about curricular opportunities for engaging students in example use, and about instructional practices related to proving and example use. A second round of analysis was descriptive and identified patterns of emergent categories and relationships. For example, teachers often noted the benefit of using "real world" examples, a benefit seemingly disconnected from example use in relation to proving-related activities. Data were coded by multiple coders, and any discrepancies were discussed until resolved.

RESULTS & DISCUSSION

In presenting the results, we first discuss teachers' views about the role examples play in proving-related activities, next we discuss teachers' expectations about students' example use, and finally we discuss teachers' evaluation of presented justifications. In each case, the survey results are not presented in their entirety, instead we present results that are representative of the majority of teachers. In addition, select teacher responses to the open-ended questions will also be provided as illustrative examples of particular findings.

1. Teachers' Views about Example Use in Proving-related Activities

A slight majority of teachers (54%) felt that students are somewhat strategic in thinking about their use of examples during proving-related activities, while about a third of the teachers (35%) felt that students were not very strategic, and the remaining teachers (11%) thought students were very strategic. When asked what purpose teachers felt students' example use served, although teachers provided a variety of responses, the most common responses were related to the role of examples in providing a concrete representation or visualization of a concept or procedure. For example, representative responses include:

> *"The purpose to which I think their use of examples serves is to visualize the problem."*
> *"Help understand the concept being taught."*
> *"Examples can guide students on what steps to getting to a certain solution are."*
> *"To illustrate the procedures for solving the problems."*
> *"They use an example that is of the exact same format as the problem at hand. Then they follow the same steps to break down/solve the problem."*

Such responses seem to speak very little about the use of examples in proving-related activities (such as testing the truth of a conjecture or exploring a conjecture), and seem to be more focused on the use of examples in general mathematical activities (such as using a worked example to understand the steps in solving an equation).

When asked whether the use of examples when developing and exploring conjectures can help students learn to develop proofs, the vast majority of teachers felt that the use of examples was helpful (43% thought example use somewhat helps, and 46% thought example use significantly helps). The majority of teachers (54%) also stated that they frequently have explicit conversations with their students about how to use and think about examples when developing, exploring, or justifying conjectures, with the remaining teachers (46%) stating that they at least occasionally have such conversations with students. Interestingly, teachers' responses regarding examples of what they might say to students appears to be less about example use during proving-related activity (with only a few exceptions) and more about general mathematical activity (e.g., following a worked example) or making connections to real life. Representative teacher responses include:

> *"Almost daily, my students are encouraged to use real world examples on as many problems as possible."*
> *"Check back to our example to find where you are making a mistake. Does your work follow the steps in the example?"*
> *"After I have assigned independent work, I usually say something similar to: 'Alright ladies and gentlemen, remember to refer back to your examples if you become stuck. They are there to assist you in working through the process.'"*
> *"Where do you see triangles in real life?"*
> *"Where would you see the graph of a quadratic function in the real world?"*
> *"We will talk about ratios and relate it to buying produce at the grocery store."*

In sum, although many teachers seemed to think that students are strategic in their example use, and that example use helps students in their proving-related activities, the nature of their open-ended responses suggests a mismatch with such views as their responses seem to focus more on examples used in non-proving related activity. It is certainly possible that teachers considered the open-ended questions more broadly (i.e., not focused specifically on proving-related example use), however, the paucity of their responses that did focus on proving-related activity is noteworthy.

3. Teachers' Expectations for Students' Example Use in Proving-related Activities

When asked about their students' approaches for exploring a provided conjecture (see Survey Question 10), half of the teachers thought it was somewhat likely that students would use examples to explore the conjecture, while almost a third (31%) thought it was very likely. Representative approaches teachers expected from their students included the following:

> *"The average student will just use one example. If they are unlucky, they may pick the one time the*

> *example works for the proofs. Getting the typical student to show two or three examples justifying their reasoning is the goal."*
> *"My students would probably find one example and decide yes/no from that one. Once I asked them to try another, they might start trying more."*
> *"They would try several different number combinations to see when it held true."*
> *"They will start testing with examples and make a table of their findings."*
> *"Trying the conjecture out with random numbers."*

When asked at what point their students would decide they were done using examples to explore the conjecture, the majority of teachers responded that students would stop using examples when they were convinced that the conjecture was true (or false), and none of the teachers mentioned anything related to moving beyond the examples to moving toward the development of a proof. Typical responses included:

> *"Once they had proven the conjecture was true or false several times."*
> *"After several attempts with different kinds of numbers, they could come to an agreement r rule about when it works and when it doesn't work."*
> *"When they were tired of proving their answer correct or when they see that the answer is constantly true. If I don't give them a minimum number of examples to use they might stop somewhere between 3-5."*
> *"I imagine them doing a few examples and stopping once they get the same result repeatedly (probably 2 or 3 attempts)."*
> *"When they probably get two in a row to justify their conclusion."*

It is interesting that the teachers' responses highlight students' example use as primarily serving to reach conviction regarding the truth of the conjecture. In other words, none of the teacher responses, with regard to typical student approaches to exploring the conjecture and when students would stop using examples, mentioned any other aspects of example use such as developing an understanding of the conjecture or gaining insight into why the conjecture is true (or false).

Teachers were also asked to indicate the various purposes students' example use might serve and the various criteria students might use in selecting their examples; in both cases the choices presented to teachers were based on the aforementioned example-use analytic framework sub-categories (Ellis et al., 2019). In the former case, teachers thought that the purposes most frequently used by students would be to check whether the conjecture is true (65%), to prove (using examples) that the conjecture is true (50%), and to explain to someone else why the conjecture is true (46%). Teachers thought that purposes of students' example use might occasionally be to disprove the conjecture (50%), to help them understand what the conjecture means (46%), and to gain insight about why the conjecture is true (42%). Interestingly, the most common response with regard to a purpose of using examples that students rarely used related to gaining insight about why the conjecture is true (46%). In the latter case, teachers thought that the most frequent criteria for example selection used by students are that the examples selected are easy to work with (54%), that the examples are the first ones that came to mind (54%), or that the examples are typical examples (39%). These criteria were also ones that teachers thought students would be more likely to use occasionally (38%, 38%, and 42%, respectively). In contrast to teachers' comments about examples occasionally serving the purpose of finding a counterexample, teachers thought that students would rarely select an example based on it being a boundary case (58%)—a criterion that often serves the purpose of searching for a counterexample.

In sum, a couple of themes in the teacher responses stand out: teachers believe that students primarily seem to use examples as a source of conviction and proof (versus other roles such as gaining insight or developing a mathematical (non-empirical) proof), and that teachers believe that students' criteria for selecting examples are often based on the ease of use of an example (likely for computing purposes) or the fact that a specific example is the first one that came to mind. Both themes stand in contrast to the ways in which examples are selected and used during productive example-use activity (Aricha-Metzer & Zaslavsky, 2019; Ozgur et al., 2019).

4. Teachers' Evaluation of Justifications based on Example Use

In this final section of results, we present teachers' thoughts regarding four justifications to a given conjecture, two empirical-based justifications and two generic example-based justifications, and, in particular, whether students would understand each approach, whether students might be likely to produce a similar approach, and whether students would be convinced by each approach. The primary intent of this series of items was to explore whether justifications based on a generic example would be viewed more favorably than justifications based on empirical evidence. The underlying rationale for the justification comparison is that generic examples tend to be viewed by mathematics educators as an especially productive as well as middle-school accessible use of examples (Leron & Zaslavsky, 2013; Zaslavsky, 2018).

In general (see *Table 1*), teachers overwhelmingly thought that the two empirically-based justifications were much more likely to be understood (95% and 91%, at least somewhat likely) and produced by students (91% and 76%, at least somewhat likely). In contrast, teachers thought that the two generic example justifications were not likely to be understood (76% and 81%, unlikely) or produced by students (100% and 95%, unlikely).

Teachers were also asked to rank order the four justifications in terms of how convincing they thought students would find each justification. Not surprisingly in light of the preceding results, teachers thought that the two empirically-based justifications would be most convincing (Approach 1: 57% most convincing, 29% second most convincing; Approach 2: 33% most convincing, 67% second most convincing), whereas they thought the two generic example justifications would be least convincing (Approach 4: 71% least convincing, 28% second least convincing; Approach 3: 24% least convincing, 62% second least convincing).

The fact that teachers did not view generic examples positively—as understandable by students, as producible by students, or as convincing to students—is somewhat surprising given, as Stylianides (2009) suggested, "generic examples are important because they can provide students with a powerful and easily reached means of conviction and explanation and can allow students to prove mathematical claims even when they lack mathematical language to express their proofs in more sophisticated ways" (p. 264). It is certainly possible that the teachers deemed the generic example justifications presented in the survey as not appropriate for middle school students, and perhaps they might have deemed other generic example justifications as more appropriate. However, the generic example justifications presented in the survey were drawn from our task-based interviews with middle school students, and thus were accessible at least to the middle school students in our previous study.

CONCLUDING REMARKS

A key conclusion from the research literature as well as from our own work is that instruction and curriculum must be *explicitly* designed to help students learn to strategically think about and productively use examples in proving-related activities (e.g., Buchbinder & Zaslavsky, 2018; Knuth, Choppin, & Bieda, 2009a; Ozgur, et al., 2019). Results from the study presented here suggest that middle school teachers do think that students are strategic in their thinking about examples, and that

Table 1. Teacher Evaluations of Justifications

	Unlikely	Somewhat Likely	Very Likely
Approach 1			
How likely is it that your students would understand this approach?	5%	57%	38%
How likely is it that your students might produce a similar response?	19%	67%	14%
Approach 2			
How likely is it that your students would understand this approach?	9%	48%	43%
How likely is it that your students might produce a similar response?	24%	67%	9%
Approach 3			
How likely is it that your students would understand this approach?	76%	14%	10%
How likely is it that your students might produce a similar response?	100%	0%	0%
Approach 4			
How likely is it that your students would understand this approach?	81%	10%	10%
How likely is it that your students might produce a similar response?	95%	5%	0%

their use of examples does support their learning to prove. Yet, teachers' responses to the open-ended response questions suggest that their instructional practices (in terms of what they say and do) are not necessarily closely aligned with their thoughts about student practices. For example, although the majority of teachers noted that they do explicitly talk with students about example use during proving-related activities, the nature of their open-ended responses suggests little focus on example use during proving-related activities but rather on example use in general mathematical practices (e.g., using worked examples to follow a solution procedure). Moreover, given the survey results are based on teachers' self-report, their responses about their own instructional practices are not always consistent with their actual instructional practices (Philipp, 2007).

Consistent with much of the literature regarding students' understandings of proof (e.g., Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009a), teachers believed that students use examples as primarily as a source of conviction and justification, and spoke little about other aspects that are associated with productive example use. Teachers also expressed that they thought students often select examples to use based on an example being easy to use or an example being the first one that came to mind—selection criteria that do not seem to consider potentially more productive mathematical selection criteria. As Iannone and colleagues (2011) suggested, "Simply asking students to generate examples about a concept [or conjecture] may not substantially improve their abilities to write proofs about that concept [or conjecture]. […] This suggests that if example generation is to be a useful pedagogical strategy, more nuance is needed in its implementation." (p. 11).

As highlighted in the previous section, the use of generic examples tends to be a method of proving thought to be accessible to students (Leron & Zaslavsky, 2013; Stylianides, 2009; Zaslavsky, 2018). Yet, at least with respect to the generic example-based justifications presented to teachers in the survey, teachers did not believe that such justifications would be accessible to students in terms of whether they would understand the arguments, whether they would be able to produce such arguments, and whether they would be convinced by such arguments. The apparent mismatch suggests that further research on student understanding of generic examples may be needed as well as that professional development opportunities may need to be designed to help teachers (re)think about generic examples as a means of justification.

Overall, the results suggest that if students are to learn to use examples productively, then teachers will need support to enact curriculum and instruction that foster such example use so that they are able to create, recognize, and capitalize on classroom opportunities to foster the development of students' strategic thinking about and productive use of examples in proving-related activities. Learning to prove is an aspect of mathematical practice that is not only notoriously difficult for students to learn and for teachers to teach, but also critically important to knowing and doing mathematics. And as Bieda (2010) suggested, "greater emphasis is needed for middle school teacher preparation, professional development, and curricular support to make justifying and proving a routine part of middle school students' opportunities to learn" (p. 380). From our perspective, this call for "teacher preparation, professional development, and curricular support" requires a re-conceptualization of research concerning students' examples-based reasoning, moving from a view of such reasoning as a stumbling block to quickly overcome toward a view of such reasoning as a necessary and critical foundation in learning to prove. Only then will curriculum and instruction enable students to learn to be strategic and deliberate in their thinking about example use, learn to use examples productively, and ultimately, to learn to prove.

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APPENDIX: Example Use Teacher Survey

Q1 **What grade level(s) do you currently teach?**

- ☐ Grade 6 (1)
- ☐ Grade 7 (2)
- ☐ Grade 8 (3)

Q2 **What mathematics course(s) do you currently teach?**

- ☐ Grade 6 Math (1)
- ☐ Grade 7 Math (2)
- ☐ Grade 8 Math (3)
- ☐ Algebra (4)
- ☐ Other (5)

Q3 **What mathematics curriculum do you currently use?**

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Q4 **How many years have you been teaching math?**

- ○ 1-3 (1)
- ○ 4-10 (2)
- ○ 10+ (3)

Q5 **When developing, exploring, or justifying conjectures, students often use examples. To what extent are students strategic in thinking about their use of examples when developing, exploring, or justifying conjectures?**

- ○ Not very strategic (1)
- ○ Somewhat strategic (2)
- ○ Very strategic (3)

Q6 **What purpose(s) do you think their use of examples serves?**

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Q7 **To what extent do you think students' use of examples when developing and exploring conjectures helps them learn to develop mathematical proofs?**

- ○ Very little help (1)
- ○ Somewhat helps (2)
- ○ Significantly helps (3)

Q8 **In your own instruction, how often do you explicitly talk with students about how to use or think about examples when developing, exploring, or justifying conjectures?**

- ○ Rarely (1)
- ○ Occasionally (2)
- ○ Frequently (3)

Q9 **Please give examples of what you might say to students.**

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Q10 ***If you add any number of consecutive whole numbers together, will the sum always be a multiple of however many numbers you added up?***

Note (for teachers): The conjecture is true for all odd numbers of consecutive numbers (for example, using 3 consecutive numbers, 3+4+5=12, and 12 is a multiple of 3), and false for all even numbers of consecutive numbers (for example, using 4 consecutive numbers, 2+3+4+5=14, and 14 is not a multiple of 4).

How likely is it that your students would use examples when exploring this conjecture?

- ○ Unlikely (1)
- ○ Somewhat likely (2)
- ○ Very likely (3)

Q11 **Describe a typical student approach to exploring and justifying the conjecture.**

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Q12 **Assuming your students would use examples, the choices below reflect potential purposes for why middle school students might use examples when exploring and justifying the conjecture. For each choice, select how often you think students use examples for that purpose.**

	Rarely (1)	Occasionally (2)	Frequently (3)
to help them understand what the conjecture states or means (1)	○	○	○
to check whether the conjecture is true (2)	○	○	○
to test a variety of cases in order to see when the conjecture is true (or false) (3)	○	○	○
to prove that the conjecture is true (e.g., examples are sufficient as proof) (4)	○	○	○
to disprove the conjecture (5)	○	○	○
to gain insight about why the conjecture is true (6)	○	○	○
to help explain (to someone else) why the conjecture is true (7)	○	○	○
to help develop a general argument that the conjecture is true (8)	○	○	○
other (please explain) (9)	○	○	○

Q13 **Assuming your students would use examples, the choices below reflect potential <u>criteria</u> for why middle school students might select examples when exploring and justifying the conjecture. For each choice, select how often you think students use that criterion.**

	Rarely (1)	Occasionally (2)	Frequently (3)
easy to work with (easy to do computations with) (1)	○	○	○
minimal starting point (starts with lowest set of numbers) (2)	○	○	○
random (chosen arbitrarily) (3)	○	○	○
boundary case (an extreme or special case) (4)	○	○	○
first thought of (first examples that came to mind) (5)	○	○	○
typical (common examples that others would likely think of) (6)	○	○	○
other (please explain) (9)	○	○	○

Q14 **Assuming your students would use examples, how do you think they would decide when they were done using examples?**

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Q15 **Consider the following (true) conjecture:** *If you add any odd number of consecutive numbers together, the sum will always be a multiple of however many numbers you added up.*

To illustrate, consider a couple examples for the cases of three and five consecutive numbers: 1+2+3=6 and 6 is a multiple of 3; or 12+13+14=39 and 39 is a multiple of 3. 1+2+3+4+5=15 and 15 is a multiple of 5; or 12+13+14+15+16=70 and 70 is a multiple of 5.

Approach 1: I tried the following examples for five consecutive numbers: 1+2+3+4+5=15 and 15 is a multiple of 5. 7+8+9+10+11=45 and 45 is a multiple of 5. 23+24+25+26+27=125 is a multiple of 5. So I think the conjecture is probably true for any five consecutive numbers but I can't be sure, and I would need to check on other cases of odd numbers of consecutive numbers to see if it's likely true for those cases.

	Unlikely (1)	Somewhat likely (2)	Very likely (3)
How likely is it that your students would understand this approach? (1)	○	○	○
How likely is it that your students might produce a similar response? (2)	○	○	○

Approach 2: I tried the following examples using different odd numbers of consecutive numbers: For 3 odd consecutive numbers: 1+2+3=6 and 6 is a multiple of 3. For 5 odd consecutive numbers: 6+7+8+9+10=40 and 40 is a multiple of 5. For 7 odd consecutive numbers: 19+20+21+22+23+24+25=154 is a multiple of 7. So I am sure the conjecture is true for the sum of all odd numbers of consecutive numbers.

	Unlikely (1)	Somewhat likely (2)	Very likely (3)
How likely is it that your students would understand this approach? (1)	○	○	○
How likely is it that your students might produce a similar response? (2)	○	○	○

Approach 3: I tried 4+5+6+7+8 and noticed that I can rewrite it as (6-2)+(6-1)+6+(6+1)+(6+2). I know the -2 & 2 and the -1 & 1 will each cancel each other, leaving five 6s, which is a multiple of 5. I know that every sequence of odd consecutive numbers has a middle number, and each sequence can also be written out and simplified in a similar way, so I am sure the conjecture is true for the sum of all odd numbers of consecutive numbers.

	Unlikely (1)	Somewhat likely (2)	Very likely (3)
How likely is it that your students would understand this approach? (1)	○	○	○
How likely is it that your students might produce a similar response? (2)	○	○	○

Approach 4: I tried 2+3+4+5+6 and noticed that I can rewrite it as 2+(2+1)+(2+2)+(2+3)+(2+4). I can then rewrite this expression as (2+2+2+2+2)+(1+2+3+4), which simplifies to 5x2+(1+2+3+4). And 5x2 is a multiple of 5 and 1+2+3+4=10 is also a multiple of 5. Since I can rewrite any sequence of five consecutive numbers this way, I know that the conjecture is always true for five consecutive numbers, however, I would need to check on other cases of odd numbers of consecutive numbers to see if it's true for those cases.

	Unlikely (1)	Somewhat likely (2)	Very likely (3)
How likely is it that your students would understand this approach? (1)	○	○	○
How likely is it that your students might produce a similar response? (2)	○	○	○

Q16 **Arrange (by dragging) the following approaches in the order that you think your students would find most convincing that the conjecture is always true. Approaches that are more convincing should be put higher up in your list.**

______ Approach 1 (1)

______ Approach 2 (2)

______ Approach 3 (3)

______ Approach 4 (4)

Q17 **Please explain your ordering of approaches.**

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