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Exploring Mathematical Reasoning of Elementary Preservice Teachers

GwiSoo Na*, Dong-Won Kim*†

*Professor, Cheongju National University of Education, South Korea

ABSTRACT

The purpose of this study is to understand the characteristics of mathematical reasoning of elementary preservice teachers (EPTs). For this purpose, 68 EPTs were presented with two tasks related to mathematical reasoning, and their written responses were analyzed. The EPTs' mathematical reasonings were investigated on the one hand as a whole and on the other hand between the three groups -the full successful, partial successful, and unsuccessful provers-, with focusing on the affordances from and strategies for using examples proposed in the CAPS framework. As the results of this study, it was revealed that the EPTs had insufficient competencies in terms of exploring with examples, finding structural features, building generalizations, developing conjectures, and producing proofs. In contrast, the EPTs had great strength with regard to representing their chosen examples in formal expression, which was a decisive contributor on the one hand but an impediment on the other in producing valid justifications to the given conjectures. Based on the results, the implications for elementary preservice teacher education were suggested.

Key Words: Mathematical reasoning, Preservice elementary teacher, Example, Conjecture, Structure, Generalization, Justification, Proof

INTRODUCTION

Mathematical reasoning is the core of mathematical practice and is crucial to all students' mathematical experiences (NCTM, 2000; Yackel & Hanna, 2003; DFE, 2013; KMOE, 2015). It is well known the mathematical reasoning and proof are one of the difficult activities which students of all educational level continue to struggle learning (Harel & Sowder, 1998; Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009; Na, 2009). The studies report that teachers, similarly to students' struggle, have a lot of difficulties in reaching enough proficiencyin mathematical reasoning and proof (Goetting, 1995: Knuth, 2002a, 2002b; Stylianides, Stylianides, & Philippou, 2007). The studies also show that teachers as well struggle to facilitate the capability of students' mathematical reasoning and proof (Peretz, 2006; Bieda, Ji, Drwencke, & Picard, 2014; Stylianides, Stylianides, & Shilling-Traina, 2013).

It is necessary for teacher educators to understand teachers' conception of mathematical reasoning and their mathematical reasoning capabilities thus help them overcome difficulties in teaching mathematical reasoning. To date, the studies on teachers' conception of mathematical reasoning and their mathematical reasoning capability have been centered on proof which is one of the main activities of mathematical reasoning, and on the middle and high school teachers (Knuth, 2002a, 2002b; Furinghetti & Morselli, 2009; Na, 2014; Lesseig, 2016; Lesseig, Hine, Na, & Boardman, 2019). In contrast, little studies for elementary school teachers' conception of mathematical reasoning and their

† corresponding author
Email: gsna21@cje.ac.kr, pourpeda@cje.ac.kr

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mathematical reasoning competency have been carried out.

The purpose of this study, in the light of the research situation, is to explore the characteristics of mathematical reasoning of elementary preservice teachers. This study centers on the main activities of mathematical reasoning such as investigating and identifying patterns, looking for structural features, making generalization, developing conjectures, and providing proofs. This study has a significance of contributing to providing a more indepth understanding of the characteristics and capabilities of mathematical reasoning of elementary preservice teachers to mathematics teacher education community, and thus providing a basis for developing elementary teacher education programs that can help them have sufficient proficiency in mathematical reasoning enough to teach elementary school students.

THEORETICAL BACKGROUND

1. Main Aspects of Mathematical Reasoning

Mathematical reasoning helps us understand why mathematical relationships exist and plays a key role in the development of an indepth understanding of mathematics (Lannin, Ellis, Elliot, & Zbiek, 2011). The importance of mathematical reasoning in school mathematics is reflected in mathematics curriculum of many countries including USA, Korea, UK, Singapore (CCSSI, 2010; DFE, 2013; KMOE, 2015; MES, 2013). In the *Common Core State Standards for Mathematics*, "Construct viable arguments and critique the reasoning of others," which is related to mathematical reasoning, is underscored as one of the Standards for Mathematical Practice. The specific content is as follows.

> Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. (CCSSI, 2010, pp. 6-7)

Mathematical reasoning is also emphasized in *Korean Mathematics Curriculum* as one of the Mathematics Subject Competencies (KMOE, 2015). Also, the teaching and learning methods to develop students' reasoning competency are suggested as follows.

> [Mathematics teachers need to] help students to conjecture mathematical facts on their own through using plausible reasoning such as induction and analogy in observation and inquiry situations and justify them based on appropriate evidence. [They should] guide students to perform logically the mathematical procedures for drawing the mathematical concepts, principles, and laws. [They also need to] encourage students to evaluate and reflect critically on whether the reasoning process is correct. (KMOE, 2015, p. 55)

A lot of studies on mathematical reasoning have been conducted in various dimensions. Since the purpose of this study is to understand the characteristics of mathematical reasoning of EPTs, we briefly review the previous studies discussing the meaning of mathematical reasoning and its major aspects.

Jeannotte and Kieren (2017) pointed out the absence of a comprehensive consensus about the term of mathematical reasoning itself and proposed a conceptual model of mathematical reasoning in school mathematics. According to them, mathematical reasoning has two central aspects, that is the *structural* and *process aspect* which are related dialectically (p. 9). The structural aspect means the form of inference such as deduction, induction, and abduction. While the process aspect refers to the various mathematical activities that occur in the process of actual inference. The process aspect embraces generalizing, conjecturing, identifying a pattern, comparing, clarifying, justifying, proving, and formal proving and exemplifying. Stylianides (2008) also suggested analytic framework of reasoning-and-proving. According to him, reasoning-and-proving is the overarching activity consisted of *making mathematical generalization* and *providing support to mathematical claims*. Here, making mathematical generalization encompass the activity of identifying patterns and making conjectures. And providing support to mathematical claims includes the activity of providing proofs and providing non-proof arguments.

Meanwhile, Lannin et al. (2011) proposed a big idea and nine essential activities of reasoning that teachers should understand to help students improve their mathematical reasoning capability. They described a big idea of mathematical reasoning as an "evolving process of conjecturing, generalizing, investigating why, and developing and evaluating arguments" (p. 10). In addition, they proposed nine essential activities of mathematical reasoning such as developing conjectures, generalizing to identify commonalities, generalizing by application, conjecturing and generalizing using terms, symbols, and representations, investigating why, justifying based on already-understood ideas, refuting a statement as false, justifying and refuting the validity of arguments, and validating justifications. Similarly, Ellis et al. (2012) suggested the essential activities related to proof such as exploring with examples, identifying patterns, looking for structural similarities across cases, making mathematical generalizations, developing conjectures, searching for counter-examples and providing proofs to mathematical claims, which can be considered major aspects of mathematical reasoning.

We, summarizing the various aspects of mathematical reasoning proposed in the previous studies above, consider the mathematical reasoning of EPTs with a focus on the aspects such as exploring with examples, identifying patterns, looking for structural features, making generalization, developing conjectures, and providing proofs in this study.

2. Elementary Teachers' Conception of Mathematical Reasoning

Several studies, although not many, investigated elementary school teachers' conception or knowledge about mathematical reasoning and proof. Looking briefly at representative studies, Martin & Harel (1989) investigated the proof frames of elementary preservice teachers. As the results, about half of the elementary preservice teachers accepted an inductive argument as a valid mathematical proof, and an incorrect deductive argument as being mathematically correct for unfamiliar statements. In addition, over a third of the elementary preservice teachers simultaneously accepted an inductive and a deductive argument as being mathematically valid. They expressed concern that if the elementary school students, by being led by elementary teachers, believe that investigating some examples - that is, inductive arguments - constitute a proof, the students would have difficulties in understanding the idea of proof in secondary school.

Melhuish, Thanheiser, & Guyot (2020) examined the elementary school teachers' noticing of justification and generalization which are essential activities of mathematical reasoning. It was found that there are discrepancies between teachers' self-reported and researchers' observed level of noticing justifying and generalizing. That is, teachers' self-reported level was higher than researchers' observed level of noticing justifying and generalizing. The researchers asserted that teachers rarely attend to justifying and generalizing in a manner consistent with the mathematics education community's view and this may account for the discrepancies.

Furthermore, several researchers suggested a framework for identifying the knowledge of elementary school teachers about mathematical reasoning and helping them teach mathematical reasoning to their students. Stylianides & Stylianides (2006), based on the findings of previous researches aiming to promote mathematics reasoning and proving in elementary school, developed a framework for the content knowledge of reasoning and proving that is helpful for teachers to teach elementary school mathematics. The framework consisted of four ideas: (a) patterns, conjectures, arguments, and proofs are connected (patterns → conjectures → arguments), (b) reasoning and proving is bounded by the community's existing knowledge, (c) mathematical definitions are central to reasoning and proving, (d) different kinds of tasks can offer different kinds of opportunities for reasoning and proving.

Herberta et al. (2015) attempted to establish a framework for understanding the growth in elementary teachers' awareness of aspects of mathematical reasoning. They, to do this, explored variation in the perceptions of mathematical reasoning held by elementary teachers. Based on the exploration, they proposed a framework consisting of seven categories of teachers' perception of mathematical reasoning along with four dimensions. The teachers' perception of mathematical reasoning such as *thinking, problem solving, validating thinking, forming conjectures, using logical arguments for validating conjectures*, *and connecting aspects of mathematics* were included as seven categories in the framework. Also, the *audience, purpose, presentation*, and *type of reasoning* constituted the four dimensions.

Meanwhile, Stylianides (2016) argued that proof, even though proof is an essential activity of mathematics, has a "marginal place" in elementary mathematics classroom internationally. He defined "proving" broadly as to denote "the mathematical activity associated with the search for a proof", and suggested the activity of proving includes a multiplicity of processes or auxiliary activities such as:

> engaging with inductive explorations to identify patterns or generalizations and make conjectures; working with particular cases or examples to test conjectures or gain a better understanding of what the conjectures mean and how they may be justified or refuted; using less formal ways of thinking (e.g., reasoning by analogy) or way representation (e.g., diagrams) to develop insight into arguments that may ultimately be developed into proofs; using rhetorical means (not necessarily mathematical) to convince others about the epistemic value of a statement (this is often referred to as argumentation. (pp. 32-33)

Furthermore, Stylianides (2016) described contributing four factors to the marginal places of a proving at elementary school such as weak knowledge about proof of many elementary teachers, their presumed believes that proving goes beyond the mathematical capability of elementary school students, the high pedagogical demand on elementary teachers trying to teach proof, and the improper and insufficient instructional support offered to elementary teachers. Many of the proving related activities mentioned above by Stylianides (2016) overlap and are consistent with the main aspect of mathematical reasoning set in this study, that is, exploring with examples, identifying patterns, looking for structural features, making generalization, developing conjectures, and providing proofs. Thus, the four factors described by Stylianides (2016) can be considered as the causes of why mathematical reasoning is not the main emphasis in elementary school.

METHODS

1. Participants and Tasks

The purpose of this study was to understand how EPTs in Korea performed a variety of essential activities of mathematical reasoning. Participants in this study were 68 EPTs, all of them were enrolled in the same university having a specific purpose for educating only elementary preservice teachers in Korea. The participants were in second or third year of the university and took 1 "liberal arts mathematics" course and 1 "elementary mathematics education" course, in both of which mathematical reasoning was not the main theme.

68 EPTs were presented with two tasks related to mathematical reasoning and asked to detail their responses to the tasks, which was to make the researchers to fully figure out the EPTs' mathematical reasoning process. The tasks, taken from Knuth et al. (2017), asked EPTs to reason whether the given conjectures on the sum and multiplication of consecutive numbers were always true and describe their reasoning process (see *Figure 1*).

[Task 1] Sungi came up with a conjecture about consecutive numbers that states, if you add any number of consecutive numbers together, the sum will be a multiple of however many numbers you added up.

(1) Do you think the conjecture will be true for any five consecutive numbers? Why?

(2) Do you think the conjecture is true for any set of consecutive numbers, not just when you pick five consecutive numbers? Why?

[Task 2] Juhee came up with a conjecture that states, if you multiply any three consecutive numbers together, the answer will be a multiple of 6. Do you think Juhee's conjecture is true? Why?

Figure 1. The consecutive numbers tasks (Knuth, Zaslavsky, & Ellis, 2017)

2. Data Analysis

The data collected and analyzed in this study were the written responses by EPTs, which included their reasoning processes to the tasks in detail. Based on the written responses, this study explored the characteristics of EPTs and qualitative differences between the subgroups of them in terms of main aspects of students' example use.

Data analysis in this study was carried out based on the Criteria, Affordances, Purposes, and Strategies (CAPS) framework described by Ellis and colleagues (Ellis et al., 2017). The CAPS framework was originally proposed as a tool to analyze students' example use in proving activity. We, although the focus of this study was on mathematical reasoning not the example use itself, applied the CAPS framework as an analytical tool since the framework encompassed the major aspects of mathematical reasoning. The CAPS framework consisted of four categories such as the criteria for choosing examples, the affordances gained from examples, the purposes for examples, and the strategies for choosing and using examples. In this study, the characteristics of mathematical reasoning of EPTs were analyzed only for the two categories of affordances and strategies among four categories in CAPS framework. The reason was that the EPTs' mathematical reasoning revealed in their written responses to the tasks were analyzed, so we could identify the affordances and strategies of them but the criteria and purposes not.

In short, the category of affordances in CAPS framework comprised the codes such as *gain insight* (*understand why, see a structural element*), *generalize*, *conjecture support* (*new conjecture, revised conjecture*), *justification support* (*viable but incomplete, produce a proof*), and *understand limitations*. With regards to the category of strategies, the strategies were again divided by two sub-categories, that is, the strategy for choosing examples and for using examples. The subcategory of strategy for choosing examples contained the codes such as *diversity*, *systematic variation* (*systematic variation-initial, systematic variation-continuation*), and *properties*. The subcategory of strategy for using examples included the codes such as *attempt to disprove, structure, improper pattern search, building formality, jumping to formality,* and informal induction.

Meanwhile, in this study, the codes appeared especially in the EPTs' responses were added to and the codes not appeared excluded from the category of affordances and strategies proposed by Ellis et al. (2017). The added codes were the *test truth* under affordances category and the *singleness* under strategies category (see *Table 1* & *Table 3*). The code of *test truth* indicated that the EPTs gained the affordance of confirming whether the conjecture given in the task was true from using their examples. The code of *singleness* referred that the EPTs chose and used only one example when they explored the conjecture. On the other hand, the excluded codes were the *understand limitations* under affordances category and *informal induction* under strategies category.

The data analysis in this study was carried out through two phases. In the first phase, all of 68 EPTs were classified into the three groups, that is, the full successful provers, the partial successful provers, and the unsuccessful provers. In the second phase, the percentage of EPTs in each group of the full successful, partial successful, and unsuccessful provers that showed the category of affordances and strategies and their sub-codes was investigated. (See *Table 1*, *Table 2*, *Table 3*, & *Table 4*). Here, the full successful provers referred to the EPTs who produced valid proofs in both [Task 1] and [Task 2], the partial successful provers to those who successfully proved only in either [Task 1] or [Task 2], and the unsuccessful provers to those who were not successful in proving in both [Task 1] and [Task 2]. In addition, the full successful provers were named the SS group and the unsuccessful provers as the US group at the convenience. The partial successful provers were named as PS group, and the partial successful provers that succeeded in proving [Task 1] and [Task 2] were named as S1 and S2, respectively.

RESULTS

In this chapter, we describe the characteristics of EPTs' mathematical reasoning on the one hand as a whole and on the other hand between the three groups, that is, the full successful (SS group), partial successful (S1 & S2 group), and unsuccessful provers (US group). The EPTs' mathematical reasonings are reported focused on the affordances from and strategies for using examples which were suggested in the CAPS framework. Among 68 EPTs who participated in this study, 22 (32.4%) were classified as the full successful, 29 (42.6%) as the partial successful, and 17 (25%) as the unsuccessful provers.

1. Affordances from example use

The details about the affordances gained from example use among the full successful, partial successful, and unsuccessful provers are presented in *Table 1* and *Table 2*. *Table 1* shows the percentages of each group among whole EPTs and *Table 2* presents the percentages of EPTs within each group. Looking at the affordances in overall, as shown in *Table 1*, 60 (88.2%) of the whole EPTs gained the affordances from using examples. The affordances that the EPTs obtained from the use of examples were: *complete proof* (37, 54.4%), *incomplete but viable proof* (14, 20.6%), *understand why* (41, 60.3%), *see a structural element* (29, 42.6%), *test truth* (15, 22.1%), *generalize* (10, 14.7%), d*evelop a*

Table 1. Distribution of affordances gained from examples among the whole EPTs

Affordances		Percentage of EPTs Frequency					Total (n=68)
		Full Successful	Partial Successful			Unsuccessful	
		SS (22, 32.4%)	PS (29, 42.7%)	S1 (24, 35.3%)	S2 (5, 7.4%)	US (17, 25%)	
Gain Insight	Understand why	21 (30.9%)	20 (29.4%)	17 (25%)	3 (4.4%)	0 (0%)	41 (60.3%)
	See a structural element	21 (30.9%)	7 (10.3%)	2 (2.9%)	5 (7.4%)	1 (1.5%)	29 (42.6%)
Generalize		8 (11.8%)	1 (1.5%)	1 (1.5%)	0 (0%)	1 (1.5%)	10 (14.7%)
Test truth		0 (0%)	7 (10.3%)	7 (10.3%)	0 (0%)	8 (11.8%)	15 (22.1%)
Conjecture support	Develop a new conjecture	6 (8.8%)	2 (2.9%)	1 (1.5%)	1 (1.5%)	1 (1.5%)	9 (13.2%)
	Revise the conjecture	4 (5.9%)	1 (1.5%)	1 (1.5%)	0 (0%)	0 (0%)	5 (7.4%)
Justification support	Incomplete but viable proof	5 (7.4%)	9 (13.3%)	4 (5.9%)	5 (7.4%)	0 (0%)	14 (20.6%)
	Complete proof	17 (25%)	20 (29.4%)	20 (29.4%)	0 (0%)	0 (0%)	37 (54.4%)
Total		22 (32.4%)	29 (42.7%)			9 (13.2%)	60 (88.2%)

new conjecture (9, 13.2%), *revise the conjecture* (5, 7.4%). Most of these affordances were from the full or partial successful provers, while test truth was mainly from the unsuccessful provers.

Looking at each group, as shown in *Table 2*, 100% of the full successful provers, 100% of the partial successful provers, and 52.9% of the unsuccessful provers gained affordances from using examples. All the full and partial successful provers obtained the affordances, while half of the unsuccessful provers gained affordances from using examples. In addition, the full or partial successful provers gained a variety of affordances, while the unsuccessful provers gained limited affordances. In the following, the characteristics of the three groups shown in each affordance will be described in more detail.

1) Justification support: Complete proof or incomplete but viable proof

Most EPTs among the full successful provers (77.2%) produced *complete proofs* through identifying formal formulas or structural properties in the context of [Task 1] and [Task 2] (see *Table 2*). These EPTs examined whether the conjecture was true to their chosen examples in [Task 1] and established algebraic expressions to produce a complete proof. They formally represented the sum of five consecutive numbers such as $a + (a + 1) + (a + 2) + (a + 3) + (a + 4) = 5a + 10$ or $(n - 2) + (n - 1) + n + (n + 1) + (n + 2) = 5n$, then they proved why the conjecture must be true. They, in [Task 2], also examined the truth or false of the conjecture with their examples and searched for the structural properties about three consecutive numbers such as

"There must be a multiple of 2 and a multiple of 3 in three consecutive numbers." Then they produced a complete proof. To illustrate, Sungwoo, one of the full successful provers, explored 2×3×4, 1×2×3, 4×5×6, 7×8×9 as examples for the product of three consecutive numbers (For reference, the names of EPTs in this study are all pseudonym). Then he identified that a number must be a multiple of 2 and a multiple of 3 to be a multiple of 6. He produced a complete proof such that the product of three consecutive numbers is a multiple of 6 because three consecutive numbers must contain a multiple of 2 and a multiple of 3. The followings were the complete proofs produced by Sungwoo in [Task 1] and [Task 2].

Table 2. Distribution of affordances gained from examples within each group

Affordances		Full Successful (n=22) SS (n=22)	Partial Successful (n=29)	S1 (n=24)	S2 (n=5)	Unsuccessful (n=17) US (n=17)	Total (n=68)
Gain Insight	Understand why	21 (95.4%)	20 (69.0%)	17 (70.8%)	3 (60%)	0 (0%)	41 (60.3%)
	See a structural element	21 (95.4%)	7 (24.1%)	2 (8.3%)	5 (100%)	1 (5.8%)	29 (42.6%)
Generalize		8 (36.4%)	1 (3.5%)	1 (4.1%)	0 (0%)	1 (5.8%)	10 (14.7%)
Test truth		0 (0%)	7 (24.1%)	7 (29.1%)	0 (0%)	8 (47.1%)	15 (22.1%)
Conjecture support	Develop a new conjecture	6 (27.2%)	2 (6.9%)	1 (4.1%)	1 (20%)	1 (5.8%)	9 (13.2%)
	Revise the conjecture	4 (18.1%)	1 (3.5%)	1 (4.1%)	0 (0%)	0 (0%)	5 (7.4%)
Justification support	Incomplete but viable proof	5 (22.7%)	9 (31.0%)	4 (16.6%)	5 (100%)	0 (0%)	14 (20.6%)
	Complete proof	17 (77.2%)	20 (69.0%)	20 (83.3%)	0 (0%)	0 (0%)	37 (54.4%)
Total		22 (100%)	29 (100%)			9 (52.9%)	60 (88.2%)

[Task 1] For example, if you add 3, 4, 5, 6, and 7, you get 25, so you divide by 5. The sum of five consecutive numbers is a + (a + 1) + (a + 2) + (a + 3) + (a + 4), which is 5a + 10, so it is divided by 5.

[Task 2] 2×3×4 = 24, 1×2×3 = 6, 4×5×6 = 120, 7×8×9 = 504. A number must be divided by 2 and 3 to be a multiple of 6. That is, it must be a multiple of 2 and a multiple of 3. Multiples of 2 appear once every 2 times, and multiples of 3 appear once every 3 times when consecutive numbers are written. Since the conjecture is said to multiply three consecutive numbers, those three consecutive numbers must contain multiples of two and multiples of three. Therefore, the conjecture is true.

Some EPTs among the full successful provers produced *incomplete but viable proofs* in [Task 1]. The following is an incomplete but viable proof suggested by Soohee, one of the full successful provers. Soohee explained that the conjecture is true for the example of 1+2+3+4+5, then yielded a valid justification by presenting a way of constructing the sum of any five consecutive numbers from 1+2+3+4+5. Soohee's justification for [Task 1] was as follows.

[Task 1] It is true. The sum of 1+2+3+4+5 is 15, which holds. No matter what five consecutive numbers we choose, it's like adding the same number to each of 1, 2, 3, 4, 5. Therefore, the sum is increased by a multiple of 5, so divided by 5. For example, in the case of 8641+8642+8643+8644+8645, 8640 is added to each term in 1+2+3+4+5, so 8640×5 is added to 1+2+3+4+5. Therefore, it is divided by 5.

In contrast, EPTs in the S1 group among the partial successful provers produced a valid proof in [Task 1] but did not in [Task 2]. Most of S1 group (83.3%), after exploring their chosen examples in both [Task 1] and [Task 2], attempted to formally represent the consecutive numbers. This attempt resulted in a complete proof in [Task 1], but it was a hindrance to build a valid justification in [Task 2].

Indeed, it is necessary to find out structural features across the examples for succeeding in [Task 2]. However, EPTs belonging to the S1 group failed to find structural features by adhering to formal expressions, and eventually did not produce a valid justification. The following is an explanation given by Eunji belonging to the S1 group. Eunji attempted to establish formal expressions in both tasks and yielded a complete proof in [Task 1] but did not in [Task 2].

> *[Task 1] 1+2+3+4+5 = 15, 2+3+4+5+6 = 20, 11+12+13+14+15 = 65, all divided by 5. When five consecutive numbers are expressed as n-2, n-1, n, n+1, n+2, the sum is 5n and thus divided by 5.*
> *[Task 2] 1×2×3 = 6, 2×3×4 = 24, 3×4×5 = 60, all of which are multiples of 6. If three consecutive numbers are n-1, n, n+1, the product of these numbers is n (n-1) (n+1) = n^3-n. It is now enough to show that n^3-n is always a multiple of 6. How can I proceed?*

EPTs belonging to the S2 group among the partial successful provers produced a complete proof in [Task 2] but did not in [Task 1]. They identified the structural factors across the examples through exploring their examples for three consecutive numbers in [Task 2]. Then they provided a complete proof why the conjecture must be true in [Task 2] which was similar to the justification by Sungwoo as seen above. On the other hand, they examined various examples and confirmed the conjecture was true but failed to provide a valid justification in [Task 1].

2) Test truth

The affordances gained from using examples of the unsuccessful provers who failed to provide a valid justification in both [Task 1] and [Task 2] were almost limited to *test truth*. They, after examining whether the conjectures were true or false to their examples, confirmed that the conjectures were true. But they failed to produce a valid justification for why the conjectures must be true. In the light of these findings, it can be said that the unsuccessful provers belonged to a kind of *naive empiricism* proposed by Balacheff (1988). The followings were the explanations by Jimin, one of the unsuccessful provers.

> *[Task 1] If the consecutive number is 1, 2, 3, 4, 5, then (1+2+3+4+5)/5 = 15/5 = 3. If the consecutive number is 2, 3, 4, 5, 6, then (2+3+4+5+6)/5 = 20/5 = 4. If the consecutive number is 3, 4, 5, 6, 7, then (3+4+5+6+7) = 25/5 = 5. Thus, the conjecture is true.*
> *[Task 2] I think it's true. After examining a few examples, I determined that the conjecture is true. For example, 1×2×3 = 6, 2×3×4 = 24, 4×5×6 = 120, these numbers are all multiples of 6.*

3) See a structural element

29 (42.6%) among the whole EPTs obtained the affordance of *see a structural element* through exploring with examples (see *Table 1*). 21 EPTs among them were the full successful provers, 7 were the partial successful provers, and 1 was an unsuccessful prover. Only 21 (30.9%) of the whole EPTs found out the structural elements in both [Task 1] and [Task 2]. Looking at within each group, 95.4% of the full successful, 24.1% of the partial successful, and 5.8% of unsuccessful provers gained the affordance of see a structural element from using examples (see *Table 2*).

Regarding the affordance of *see a structural element*, a notable result was that only 8.3% among S1 group found structural features through using examples. EPTs belonging to S1 group were who succeeded in proving the conjecture in [Task 1] but not in [Task 2]. In fact, identifying the structural elements across examples by examining them for the product of three consecutive numbers plays a decisive role in making a proof in [Task 2]. In other words, it is important to discover two structural features, that is, a number must be a multiple of 2 and a multiple of 3 in order to be a multiple of 6, and there must be one number that is a multiple of 2 and another number that is a multiple of three among three consecutive numbers. The EPTs belonging to S1 group failed to find structural features and eventually failed to produce a proof for the conjecture in [Task 2]. Most EPTs in S1 group algebraically expressed the product of three consecutive numbers such as n×(n+1)×(n+2) = n^3+3n^2+2n or (a-1)×a×(a+1) = a^3–a in [Task 2], similar to Eunji illustrated earlier. They attempted to yield a valid justification through manipulating these formal expressions, but eventually did not succeed in making a valid justification.

4) Generalize

Only 10 (14.7%) among the whole EPTs gained the affordance of generalize from using examples. These EPTs all made general propositions in [Task 1], but no EPTs attempted to make generalization in [Task 2]. Looking at each group, 36.4% of the full successful, 4.1% of the partial successful, and 5.8% of the unsuccessful provers gained *generalize* affordance from using their examples. Meanwhile, all EPTs who made generalization developed new or revised conjectures based on their generalization. It can be asserted that making generalization through exploring examples would be the basis for developing new conjectures.

The ways EPTs made generalization were divided into three types. The first way was to make a generalized proposition depending on the result whether the conjecture was true or false for a few examples. This way of generalization can be described as a kind of *empirical generalization* (Bills & Rowland, 1999) or *result pattern generalization* (RPG) (Harel, 2001), in that it focused on the regularity in the results whether the conjecture was true for some examples. The following generalization made by Jinju can be described as a kind of *empirical generalization*. Jinju found that the sum of three, five, and seven consecutive numbers divided by 3, 5, and 7, respectively. Then she made a generalized proposition from this result such as "The conjecture is true for odd consecutive numbers." The generalization made by Jinju was as following:

1+2+3 = 6. 6÷3 = 2; 1+2+3+4+5 = 15. 15÷5 = 3; 1+2+3+4+5+6+7 = 28. 28÷7 = 4; .... Thus, the conjecture is true for odd consecutive numbers.

The second way was to create a generalization through finding the structural features in the sum of five consecutive numbers and extending it to the sum of odd consecutive numbers. This way of generalization can be described as a kind of *structural generalization* (Bills and Rowland, 1999) in that it focused on the underlying structure across odd consecutive numbers with exploring some examples. To illustrate, Minsoo's generalization can be described as a type of *structural generalization*. Minsoo found a structure that five consecutive numbers "symmetrically exist" around the center number. Then he made a generalization by expanding the structure to odd consecutive numbers and symmetrically expressing the sum of them such as (a-n) + (a-(n-1)) ⋯ + a + ⋯ + (a+ (n-1)) + (a+n).

(1) *1 + 2 + 3 + 4 + 5 = 15, 15 ÷ 5 = 3. 2 + 3 + 4 + 5 + 6 = 20, 20 ÷ 5 = 4. The five consecutive numbers are symmetric about the middle number. If we put the middle number in five consecutive numbers as x (x is a natural number of 3 or more), then the numbers can be set as x-2, x-1, x, x+1, x+2. (x-2) + (x-1) + x + (x+1) + (x+2) = 5x. 5x÷5 = x. x is a natural number of 3 or more, so the sum of five consecutive numbers is divided by 5.*

(2) *Any odd consecutive numbers are symmetric about the middle number, as in five consecutive numbers. If we put the middle number in odd consecutive numbers as x, then the sum is (x-n) + (x-(n-1)) + ⋯ + x + ⋯ + (x+(n-1)) + (x+n) = (2n+1) x. Since (2n+1) x ÷ (2n+1) = x and x is a natural number, this is divisible. Therefore, the conjecture is always true for any odd consecutive numbers.*

The third was to make generalization through representing the sum of consecutive numbers to formal expressions and algebraically operating the formal expressions. The authors propose to call this generalization as *formal generalization*. To illustrate, the generalization made by Yunwoo can be described as a type of formal generalization. Yunwoo expressed formally the sum of n consecutive numbers as (x+1) + (x+2) + ⋯ + (x+n) = nx + {n(n+1)}/2 after investigating whether the conjecture is true or false for some examples. And he performed algebraic manipulations such as [nx + {n(n+1)/2}] ÷ n = x + (n+1)/2. Then he developed a generalized proposition that the conjecture was true for odd consecutive numbers, because n must be odd in order for (n+1)/2 to be a integer. The generalization Yunwoo made was like this:

1+2 = 3, 3÷2 = 3/2; 1+2+3 = 6, 6÷3 = 2; 1+2+3+4 = 10, 10÷4 = 5/2; 1+2+3+4+5 = 15, 15÷5 = 3; 1+2+3+4+5+6 = 21, 21÷6 = 7/2.
sum of n consecutive numbers:
(x+1) + (x+2) + ⋯ + (x+n) = nx + (1+2+3+⋯ + n) = nx + {n (n+1)/2}

[nx+{n (n+1)/2}] ÷ n = x + (n+1)/2.
Therefore, the conjecture is always true when n+1 is even, that is, n is odd.

Among the 10 EPTs who made generalization, only Jinju who was an unsuccessful prover showed *empirical generalization*. The other 9 EPTs, who were the full or partial successful provers, made generalization characterized by *structural generalization* or *formal generalization*. It seems, from these results, that *structural generalization* or *formal generalization* provides more insight into why the conjecture must be true and thus helps to produce proofs than *empirical generalization*.

5) Conjecture support: Develop a new conjecture or revise the conjecture

Only 14 (20.6%) among the whole EPTs obtained the affordance of *conjecture support* from using examples. These EPTs all developed new or revised conjectures only in [Task 1], none in [Task 2]. 10 EPTs among them were the full successful, 3 EPTs were the partial successful, and 1 EPT was an unsuccessful prover. In addition to, 9 (13.2%) EPTs developed new conjectures and 5 (7.4%) EPTs produced revised conjectures. Looking at each group, 45.3% of the full successful, 10.3% of the partial successful, and 5.8% of unsuccessful provers gained *conjecture support* affordance from using examples.

EPTS who developed new conjectures investigated together examples of even consecutive numbers as well as examples of odd consecutive numbers. Then they developed new conjectures such as "the conjecture was true for odd but false for even consecutive numbers." In contrast, some of EPTs who attempted to improve the given conjectures developed revised conjectures like "the sum of odd consecutive numbers is divided by the odd number" after confirming whether the given conjecture was true for their examples of odd consecutive numbers. Other of those developed revised conjectures such as "the sum of even consecutive numbers is divided by the even number." through using their examples of even consecutive numbers as counterexamples.

2. Strategies for choosing and using examples

The distributions of percentage of the EPTs among whole group and within each group who used a given strategy are presented in *Table 3* and *Table 4*, respectively. Looking at in overall, as shown in *Table 3*, 63 (92.6%) of the whole EPTs employed the strategies for choosing examples. Among them, 22 (32.4%) were the full successful, 27 (39.8%) were the partial successful, and 14 (20.6%) were the unsuccessful provers. In terms of the strategies for using examples, 56 (82.4%) of the whole EPTs employed a variety of strategy for using examples. Among them, 21 (30.9%) were the full successful, 25 (36.8%) were the partial successful, and 10 (14.7%) were the unsuccessful provers. The results revealed that the EPTs were more strategic with regards to choose than to use examples.

Looking at within each group, as shown in *Table 4,* 100% of the full successful, 93.1% of the partial successful, and 82.4% of the unsuccessful provers employed the strategies for choosing examples. In terms of the strategies for using examples, 100% of the full successful, 86.2% of the partial successful, and 58.8% of unsuccessful provers employed the strategies for using examples. Comparison of the full successful, partial successful, and unsuccessful provers showed that the differences between them were greater in the strategies for using (100% versus 86.2% versus 58.8%) than for choosing examples (100% versus 93.1% versus 82.4%). In particular, the difference between the unsuccessful provers and the other two groups was large in terms of the strategies for using examples.

Regarding the difference between the percentage of EPTs who employed the strategies for choosing and for using examples within each group, there was no difference for the full successful (100% versus 100%) and a little difference for the partial successful provers (93.1% versus 86.2%). However, the difference was quite large in the case of the unsuccessful provers (82.4% versus 58.8%). It seems that the full or partial successful provers showed almost same high degree of strengths in both strategies for choosing and for using examples. On the other hand, the unsuccessful provers were quite strategic about how to choose examples, but they lacked the strategies for how to use examples productively. In the following, the characteristics of the three groups shown in each strategy will be described in more detail.

1) Strategies for choosing examples

The strategies for choosing examples preferred by the three groups showed commonalities and differences at the same time. We consider the characteristics between the three groups and within each group in terms of each strategy for choosing examples in more detail below.

Table 3. Distribution of the strategies for example use among the whole EPTs

Strategies	Percentage of EPTs Frequency				
	Full Successful	Partial Successful		Unsuccessful	Total (n=68)
	SS (22, 32.4%)	PS (29, 42.7%)		US (17, 25%)	
		S1 (24, 35.3%)	S2 (5, 7.4%)		
Strategies for example choice	**22 (32.4%)**	**27 (39.8%)**		**14 (20.6%)**	**63 (92.6%)**
		22 (32.4%)	**5 (7.4%)**		
Singleness	7 (10.3%)	9 (13.2%)		1 (1.5%)	17 (25%)
		7 (10.3%)	2 (2.9%)		
Diversity	2 (2.9%)	3 (4.4%)		3 (4.4%)	8 (11.8%)
		2 (2.9%)	1 (1.5%)		
Systematic variation-Initial	12 (17.6%)	18 (26.5%)		12 (17.6%)	42 (61.8%)
		15 (22.1%)	3 (4.4%)		
Systematic variation-Continuation	4 (5.9%)	4 (5.9%)		1 (1.5%)	9 (13.2%)
		2 (2.9%)	2 (2.9%)		
Properties	3 (4.4%)	4 (5.9%)		4 (23.5%)	11 (16.2%)
		2 (2.9%)	2 (2.9%)		
Strategies for example use	**21 (30.9%)**	**25 (36.8%)**		**10 (14.7%)**	**56 (82.4%)**
		20 (29.4%)	**5 (7.4%)**		
Attempt to disprove	12 (17.6%)	16 (23.5%)		10 (14.7%)	38 (55.9%)
		13 (19.1%)	3 (4.4%)		
Structure	21 (30.9%)	7 (10.3%)		1 (1.5%)	29 (42.6%)
		2 (2.9%)	5 (7.4%)		
Improper pattern search	0 (0%)	6 (8.9%)		5 (7.4%)	11 (16.2%)
		5 (7.4%)	1 (1.5%)		
Building formality	20 (29.4%)	19 (28.0%)		0 (0%)	39 (57.4%)
		19 (28.0%)	0 (0%)		
Jumping to formality	7 (10.3%)	21 (30.9%)		8 (11.8%)	36 (52.9%)
		20 (29.4%)	1 (1.5%)		
TOTAL	**22 (32.4%)**	**27 (39.7%)**		**14 (20.6%)**	**63 (92.6%)**
		22 (32.4%)	**5 (7.4%)**		

The most preferred strategy for choosing examples by the full successful, partial successful and unsuccessful provers was the *systematic variation-initial* in common. The EPTs who employed the

systematic variation-initial strategy systematically chose a set of examples such that they shifted the nature of each successive example by varying one or more elements, as shown in Minsoo's examples. Minsoo chose {1, 2, 3, 4, 5} and {2, 3, 4, 5, 6} as his examples to explore the conjecture, in which the second example {2, 3, 4, 5, 6} was chosen by changing the element 1 in the first example {1, 2, 3, 4, 5}. It seems that, based on the relatively large number of EPTs (61.8%) employing the *systematic variation-initial* strategy, EPTs had a tendency for choosing new examples by systematically changing the previous example.

The three groups of EPTs had in common that they preferred most for the *systematic variation-initial* strategy, but they showed a big difference in the way they used their examples chosen by the strategy. In the case of unsuccessful provers, a relatively large number of them (70.5%) chose examples by the strategy, while they failed in producing proofs. In addition, only 52.9% gained affordances from using their examples, in which even

Table 4. Distribution of the strategies for example use within each group

Strategies	Percentage of EPTs Frequency					
	Full Successful (n=22)	Partial Successful (n=29)			Unsuccessful (n=17)	Total (n=68)
	SS (n=22)		S1 (n=24)	S2 (n=5)	US (n=17)	
Strategies for example choice	**22 (100%)**	**27 (93.1%)**	**22 (91.7%)**	**5 (100%)**	**14 (82.4%)**	**63 (92.6%)**
Singleness	7 (31.8%)	9 (31.0%)	7 (29.1%)	2 (40%)	1 (5.8%)	17 (25%)
Diversity	2 (9.1%)	3 (10.3%)	2 (8.3%)	1 (20%)	3 (17.6%)	8 (11.8%)
Systematic variation-Initial	12 (54.5%)	18 (62.1%)	15 (62.5%)	3 (60%)	12 (70.5%)	42 (61.8%)
Systematic variation-Continuation	4 (18.1%)	4 (13.8 %)	2 (8.3%)	2 (40%)	1 (5.8%)	9 (13.2%)
Properties	3 (13.6%)	4 (13.8%)	2 (8.3%)	2 (40%)	4 (23.5%)	11 (16.2%)
Strategies for example use	**21 (95.4%)**	**25 (86.2%)**	**20 (83.3%)**	**5 (100%)**	**10 (58.8%)**	**56 (82.4%)**
Attempt to disprove	12 (54.5%)	16 (55.2%)	13 (54.1%)	3 (60%)	10 (58.8%)	38 (55.9%)
Structure	21 (95.4%)	7 (24.1%)	2 (8.3%)	5 (100%)	1 (5.8%)	29 (42.6%)
Improper pattern search	0 (0%)	6 (20.7%)	5 (20.8%)	1 (20%)	5 (29.4%)	11 (16.2%)
Building formality	20 (90.9%)	19 (65.5%)	19 (79.1%)	0 (0%)	0 (0%)	39 (57.4%)
Jumping to formality	7 (31.8%)	21 (72.4%)	20 (83.3%)	1 (20%)	8 (47.1%)	36 (52.9%)
TOTAL	**22 (100%)**	**27 (93.1%)**	**22 (91.7%)**	**5 (100%)**	**14 (82.4%)**	**63 (92.6%)**

the affordances were biased on test truth. In other words, the unsuccessful provers who adopted the strategy could choose their examples in a systematic way but failed to use their examples productively and then stopped in checking whether the given conjecture was true or not for their examples. In contrast, the full or partial successful provers adopting the *systematic variation-initial* strategy succeeded in making valid justifications through using productively as well as choosing systematically their examples.

This result implies that the strategy for using examples is more closely related to making valid justifications than that of choosing examples. That is, the choice of examples in a systematic way by EPTs itself does not guarantee their production of valid justifications. We contend that, with regards to the EPTs' constructing valid justifications, their strategy for using examples would be more important factor than for choosing examples.

Meanwhile, the strategy for choosing examples that differed greatly between the full or partial successful provers and unsuccessful provers was the *singleness* strategy. EPTs who employed the *singleness* strategy chose only one example to explore the given conjecture. In the case of full or partial successful provers, about 31% of each chose example by the strategy. They were good at using the single example as a generic example (i.e., seeing generality through a particular example). Most of them identified the structural elements and generality on the sum of five consecutive numbers or the multiplication of three consecutive numbers through investigating a single example such as {1, 2, 3, 4, 5} or {2, 3, 4}. In contrast, 5.8% (only one) among unsuccessful provers chose example by the *singleness* strategy. In addition to, the unsuccessful prover did not attempt to use his one example to produce a valid justification, only confirming whether the given conjecture was true or false for the example.

Another noteworthy result was that a relatively high percentage (23.5%) of unsuccessful provers employed the properties strategy, unlike the full or partial successful provers (13.6% and 13.8%). The unsuccessful provers who employed the properties strategy explored examples of consecutive numbers including negative numbers such as {-2, -1, 0, 1, 2} or {-1, 0, 1}, and claimed the conjecture was not true. That is, they insisted that the conjecture "the sum of any five consecutive numbers is a multiple of five" was not always true because $-2 + (-1) + 0 + 1 + 2 = 0$ and 0 was not a multiple of five. Also, since $(-1) \times 0 \times 1 = 0$ and 0 is not a multiple of 6, it was argued that the conjecture "the product of any three consecutive numbers is a multiple of 6" was false. These unsuccessful provers had a mathematical misconception that 0 was not a multiple of any number, and such misconception hindered their valid mathematical reasoning. We assert, based on these results, that the valid mathematical reasoning of EPTs depends entirely on their understanding about mathematical concepts.

2) Strategies for using examples

Looking at the strategies for using examples in overall, as shown in *Table 4*, EPTs used their chosen examples mainly to *build formality* (57.4%), *attempt to disprove* (55.9%), *jump to formality* (52.9%), and *look for structure* (42.6%). Looking at the strategies more closely, there were the commonalities and differences among the full successful, partial successful, and unsuccessful provers. We consider the characteristics between and within the three groups in terms of each strategy for using examples in more detail below.

The strategy for using examples which the full successful, partial successful and unsuccessful provers had similar preferences was the *attempt to disprove* (54.5% versus 55.2% versus 58.8%). The EPTs who adopted this strategy were good at insisting that the conjecture was not true through using their examples as counterexamples. To illustrate, Yunji, one of the unsuccessful provers, chose her example of {1, 2} and argued that the conjecture was not always true because $1+2 = 3$ and 3 was not a multiple of 2 in [Task 1]

The most notable strategies for using examples between three groups were those related to formality. In all three groups, a lot of EPTs showed the strategies in common to formally represent consecutive numbers through exploring with their examples. On closer examination, however, there were significant differences between three groups in terms of how they used each strategy related to formality. First, the full successful and unsuccessful provers showed opposite tendency in related to the strategy of *building formality* and *jumping to formality*. Most of the full successful provers (90.9%) employed the *building formality* strategy, but a relatively small number of them (31.8%) employed the *jumping to formality* strategy. In contrast, among the unsuccessful provers, no EPTs employed the *building formality* strategy (0%), but a relatively large number of them (47.1%) employed the *jumping to formality* strategy. To add an explanation, the *building formality* strategy made a decisive role to producing valid justifications for the given conjecture, but *the jumping to formality* strategy did not help at all in generating valid justifications. The *building formality* strategy can be seen in Sungwoo's reasoning mentioned above. Sungwoo, one of the full successful provers, used his chosen examples to build a formal expression such as $a + (a+1) + (a+2) + (a+3) + (a+4)$ for the sum of five consecutive numbers, and then succeeded in producing a complete proof in [Task 1]. In contrast, Hojung, one of the unsuccessful provers, jumped to represent the product of three consecutive numbers in an algebraic expression such as $n \times (n+1) \times (n+2) = n^3 + 3n^2 + 2n$ from her examples, but did not produce a valid justification for the conjecture in [Task 2]. Hojung's explanation was as follows:

> *[Task 2]* $1\times2\times3 = 6$, $6\div6 = 1$; $2\times3\times 4 = 24$, $24\div6 = 4$; $10\times11\times12 = 1320$, $1320\div6 = 220$. *When we put three consecutive numbers as n, n+1, n+2, the product of these numbers is* $n \times (n+1) \times (n+2) = n^3 + 3n^2 + 2n$. *Now we need to show* $n^3 + 3n^2 + 2n$ *is a multiple of 6.*

Meanwhile, among the partial successful provers, the proportion who employed the *building formality* or *jumping to formality* strategy was similar (65.5% versus 72.4%). In addition, a lot of partial successful provers belonging to S1 group employed both the strategy of *building formality* in [Task 1] and *jumping to formality* in [Task 2]. Eunji's explanation mentioned earlier shows both the *building formality* and *jumping to formality* strategy. Eunji built the algebraic expression such as (n-2) + (n-1) + n + (n+1) + (n+2) for the sum of five consecutive numbers through exploring her examples in [Task 1], and succeeded in producing a complete proof. On the other hand, Eunji employed the strategy of *jumping to formality* in which the product of three consecutive numbers was expressed as n (n-1) (n+1) = n^3- n, but did not succeed in producing a valid justification in [Task 2].

With regards to the strategy of *structure*, the proportion of EPTs who adopted the strategy between full successful, partial successful, and unsuccessful provers differed significantly. While 95.4% of the full successful provers adopted the *structure* strategy, the partial successful and unsuccessful provers who adopted the strategy were only 24.1% and 5.8%, respectively. It seems that whether EPTs looked for the structural features across their examples was an important factor in determining which of the three groups they belonged to. Indeed, the EPTs needed to find out the structural features across examples about the multiplication of three consecutive numbers for succeeding in proving the conjecture given in [Task 2]. Most of the full successful provers produced a complete proof by finding out the structural features such as "Any three consecutive numbers have one multiple of 2 and one multiple of 3. Therefore, the multiplication of them must be a multiple of 6." as seen in Sungwoo's explanation mentioned earlier. In contrast, only 17.2% of the partial successful provers, who belonged to the S2 group, looked for structural features across their examples and then succeeded in proving the conjecture given in [Task 2]. On the other hand, 82.8% of the partial successful provers, who belonged to the S1 group, did not succeed in producing proofs because they did not find out structural features as seen in Eunji's explanation. Among the unsuccessful provers, only 5.8% (only 1 EPT) adopted the strategy of *structure*.

DISCUSSION AND IMPLICATIONS

We discuss the main characteristics of EPTs' mathematical reasoning in terms of developing conjectures, making generalization, finding structural features, and generating proofs through exploring with examples in this chapter. Furthermore, we consider the implications for elementary teacher education that can be derived from the results of this study.

1. Discussion

EPTs in overall showed insufficient competences with regards to investigating and using examples, developing conjectures, and making mathematical generalizations which are the main aspects of mathematical reasoning. Conjecturing is meant to reason regular mathematical properties through finding similarities and differences between various examples (Jeannotte & Kieran, 2017). Only 14 (20.6%) of the whole EPTs developed new or revised conjectures through investigating their chosen examples (See *Table 1*). They developed their own conjectures through finding structural features across their examples for odd consecutive numbers and using their examples for even consecutive numbers as counterexamples in [Task 1]. They developed revised or new conjectures such as "the conjecture is true for any odd consecutive numbers," "the conjecture is false for even consecutive numbers," and "the conjecture is true for odd but false for even consecutive numbers." On the other hand, a significant number (52, 79.4%) of EPTs did not develop any conjectures.

In terms of generalization, only 10 (14.7%) EPTs made generalization by exploring their chosen examples (See *Table 1*). Most of the EPTs (86.8%) did not attempt to generalize. Making generalization is essential to mathematical reasoning (Stylianides, 2008; Artzt, 1999). According to Stylianides (2008), making generalization is meant to "the transportation of mathematical relations from given sets to new sets for which the original sets are subsets" (p. 9). In this study, it is possible to make a generalization such as "the sum of odd(n) of consecutive numbers is divided by the odd number(n)" from the given conjecture that "the sum of 5 consecutive numbers is divided by 5" in [Task 1]. Also, in [Task 2], the generalized proposition "the product of m consecutive numbers is a multiple of m!" can be derived from the given conjecture "the product of three consecutive numbers is a multiple of 6." All of 10 who made generalization developed generalized propositions in [Task 1]. Furthermore, all of them produced new or revised conjectures. It seems that making generalization through exploring with examples would be the basis for developing conjectures.

Regarding to producing proofs, EPTs showed the tendency to prefer and adhere to proving through algebraic and formal expressions. EPTs' attempts to prove through algebraic expressions had both advantages and disadvantages. EPTs' proving through algebraic expressions played a decisive role in producing valid proofs, but on the one hand, it was a limiting factor in producing proofs. Indeed, representing the sum of five consecutive numbers in the formal expressions such as n + (n+1) + (n+2) + (n+3) + (n+4) = 5n or (a -2) + (a -1) + a + (a+1) + (a+2) = 5a played a critical role in producing a complete proof for why the conjecture must be true in [Task 1]. 39 (57.4%) EPTs who build such algebraic expressions and employed *building formality* as a strategy for using examples were succeed in proving the conjecture in [Task 1]. In contrast, representing the product of three consecutive numbers in algebraic forms such as $n \times (n+1) \times (n+2) = n^3 + 3n^2 + 2n$ or $(a-1) \times a \times (a+1) = a^3-1$ did not help to produce proofs about the conjecture in [Task 2]. 28 (41.2%) EPTs established such algebraic expressions in [Task 2], but they could not go further from their algebraic expression and eventually did not produce a proof. We, from these results, assert that EPTs' building algebraic expressions is not always helpful in producing proofs for why the conjecture must be true.

In line with the above discussion, it was important for EPTs to identify the structural features across examples for succeeding in producing valid justification to the conjecture in [Task 2]. That is, the EPTs had to find structural features such that "it must be a multiple of 2 and at the same time a multiple of 3 in order to be a multiple of 6," and "any three consecutive numbers have one multiple of 2 and one multiple of 3." Only 27 (39.7%) EPTs who identified these structural features and employed structure strategy for using examples succeeded in producing proofs in [Task 2]. The other 41 (60.3%) EPTs did not find the structural features across their examples and were not successful in producing proofs in [Task 2]. According to Ellis, Knuth, & Bieda (2012), "looking for structural similarities across cases" is one of the important activities needed in the process of successful proof. In addition, identifying structural features or regularity by searching for similarities between examples is important as a process aspect of mathematical reasoning (Jeannotte & Kieran, 2017). The result that a lot of EPTs did not search for structural features look through examples suggest that EPTs need to learn how to find structures with using examples.

EPTs' preference and adherence to algebraic expressions was reflected in the success rate of proving in [Task 1] and [Task 2]. EPTs showed a greater success rate of proving in [Task 1] than in [Task 2]. 46 (67.7%) EPTs were successful in [Task 1], while 27 (39.8%) EPTs were successful in proving the conjecture in [Task 2] (see *Table 1*). In addition, only 22 (32.4%) EPTs successfully proved the conjectures in both [Task 1] and [Task 2]. Meanwhile, Hanna (2000) distinguished between *proof that proves and proof that explains*. In reference with Hanna (2000), EPTs' proof such as "Any five consecutive numbers can be set to n, n + 1, n + 2, n + 3, n + 4. Then the sum is 5n + 10, so it is divided by 5." to the conjecture in [Task 1] corresponds to a kind of *proof that proves*. On the other hand, EPTs' following proof can be described as a kind of *proof that explain*: "A number must be a multiple of 2 at the same time a multiple of 3 in order to be a multiple of 6. Any three consecutive numbers have one multiple of 2 and one multiple of 3. Therefore, the product of three consecutive numbers is always a multiple of 6." In the light of EPTs' success rate of proving in [Task 1] and [Task 2], it can be said that they have more powerful reasoning ability in terms of *proof that proves* than *proof that explains*.

Meanwhile, 17 unsuccessful provers who were accounted for one-fourth of whole EPTs perceived confirming examples as valid justification but did not attempt to do any justification. They insisted that the conjecture was always true merely by testing it for some examples they chose. They did not proceed beyond verifying the conjecture to understanding why it must be true. These EPTs need to learn what a valid justification is and what it should be equipped with, along with why confirming examples is deficient.

2. Implications

EPTs in Korea were found to obtain some limited affordances from and to be insufficient in their strategy for using examples. EPTs were somewhat insufficient in finding structural features, building generalizations, developing conjectures, and producing justifications through exploring with examples. In contrast, EPTs showed great strength in representing the conjecture into algebraic expressions through investigating examples.

Exploring examples, finding structural features, making generalization, developing conjectures, and generating justifications are essential aspects of mathematical reasoning activities (Ellis, Bieda, &

Knuth, 2012; Jeannotte & Kieran, 2017; Stylianides, 2008). Insufficient mathematical reasoning competency that the EPTs participated in this study revealed in navigating with examples, searching for structural elements, building generalizations, constructing conjectures, and producing justifications gives a lot of implications for elementary teacher education. EPTs' insufficient capability in mathematical reasoning will be likely to be a limiting element for them to instruct mathematical reasoning for elementary school students when they will teach in the future. It would be difficult for the teachers who do not fully perform mathematical reasoning to guide students' mathematical reasoning faithfully. But it is unfortunate to dismiss EPTs' insufficient competency in mathematical reasoning simply by their defect because they are not mathematics specialist (Stylianides, 2016). It is necessary for teacher educators to provide teacher education programs in which EPTs can improve their insufficient mathematical reasoning competency. Teacher educators need to help EPTs experience a variety of mathematical reasoning activities including exploring examples, searching for structures, building generalizations, and developing and justifying mathematical conjecture through the programs.

Meanwhile, considering the cognitive development of elementary school students whom EPTs will teach in the future, they are expected to be in the *stage of concrete operation* (Inhelder & Piaget, 1958) and have difficulties in formal reasoning. Thus, it would be more appropriate to teach the students to use examples productively for searching for structural features, making generalization, developing conjectures, and constructing justifications. EPTs need to understand and experience the appropriate ways to teach mathematical reasoning for the students who are in the *concrete operation* stage. In order to support better the mathematical reasoning of the students, EPTs should be able to perform mathematical reasoning with using examples and recognize their significance. We therefore assert that teacher educators need to continually encourage EPTs to investigate examples, use examples productively, look for patterns between examples, and justify using structural features across examples.

EPTs' preference to represent the conjecture in formal expression played a decisive role in producing valid justifications, while on the other hand, was an obstacle for making valid justifications. EPTs need to learn and appreciate the importance of flexible use of formal expressions and the limitations of excessive algebraization, along with the advantages of formal expression. To illustrate, EPTs should understand that expressing the conjecture given in [Task 2] as an algebraic form such as $x(x+1)(x+2) = 6p$ and sticking only to the algebraic form may be a barrier to yielding valid justification. In addition, for elementary school students, it would be more adequate to instruct mathematical justification by finding structures through exploring examples rather than by making formal expressions. Therefore, teacher educators need to provide EPTs with a variety of tasks related to mathematical reasoning, in which EPTs can learn how to apply algebraic and formal expressions to the characteristics of tasks.

References

Artzt, A. F. (1999). Mathematical reasoning during small-group problem solving. In L.V. Stiff (Ed.), *Developing mathematical reasoning in grades K-12* (1999 Yearbook, pp. 115–127). Reston, VA: NCTM.

Balacheff, N. (1988). Aspects of proof in pupil's practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers and children*. London: The Open University, 216-235.

Bieda, K., Ji, X., Drwencke, J., & Picard, A. (2014). Reasoning-and-proving opportunities in elementary mathematics textbooks. *International Journal of Educational Research*, *64*, 71–80.

Bills, L., & Rowland, T. (1999). Examples, generalization and proof. In L. Brown (Ed.), *Making meaning in mathematics. Advances in mathematics education* (pp. 103–116). York, UK: QED.

CCSSI (2010). *Common core state standards for mathematics*. Common Core State Standards Initiative, http: //www. corestandards.org

DFE (2013). *The national curriculum in England: Key stages 3 and 4 framework*. Department for Education: Retrieved from https://www.gov.uk/government/publications/national-curriculum-in-england-secondary-curriculum.

Ellis, A., Bieda, K., & Knuth, E. (2012). *Developing essential understanding of proof and proving for teaching mathematics in grades 9–12*. Reston, VA: National Council of Teachers of Mathematics.

Ellis, A., Ozgur, Z., Vinsonhaler, R., Dogan, M. F., Carolan, T., Lockwood, T., Lynch, A., Sabouri, P., Knuth, E., & Zaslavsky, O. (2017). Student thinking with examples: The criteria-affordances-purposes-strategies framework. *Journal of Mathematical Behavior*, http://dx.doi.org/10.1016/j.jmathb.2017.06.003

Furinghetti, F., & Morselli, F. (2009). Teachers' beliefs and the teaching of proof. In Fou-Lai Lin, Feng-Jui Hsieh, Gila Hanna, & Michael de Villiers (Eds.), *Proceedings of the ICMI Study 19 conference: Proof and Proving in Mathematics Education*. Taipei, Taiwan: National Taiwan Normal University, 166-171

Goetting, M. (1995). The college students' understanding of mathematical proof (Doctoral dissertation, University of Maryland). *Dissertations Abstracts International*, *56*, 3016A.

Hanna, G. (2000). Proof: explanation and exploration: An overview. *Educational Studies in Mathematics*, *44*, 5–23.

Harel, G. (2001). The development of mathematical induction as a proof scheme: A model for DNR-based instruction. In S. Campbell, & R. Zazkis (Eds.), *Learning and teaching number theory: Research in cognition and instruction* (pp. 185–212). Norwood, NJ: Ablex Publishing Corporation.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education III* (pp. 234-283). Washington, DC: Mathematical Association of America.

Herberta, S., Valeb, C., Braggb, L. A., Loongb, E., & Widjajab, W. (2015). A framework for primary teachers' perceptions of mathematical reasoning. *International Journal of Educational Research*, *74*, 26-37.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education*, *31*(4), 396–428.

Inhelder, B. & Piaget, J. (1958). *The growth of logical thinking: from childhood to adolescence*. London: Routledge & Kegan Paul Ltd.

Jeannotte, D., & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. *Educational Studies in Mathematics*, *96*, 1–16.

KMOE (2015). *2015 revised Korea mathematics curriculum*. Korea Ministry of Education.

Knuth, E. (2002a). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education*, *33*(5), 379–405.

Knuth, E. (2002b). Teachers' conceptions of proof in the context of secondary school Mathematics. *Journal of Mathematics Teacher Education*, *5*, 61-88.

Knuth, E., Choppin, J., & Bieda, K. (2009). Middle school students' production of mathematical justifications. In D. Stylianou, M. Blanton, & E. Knuth (Eds.), *Teaching and learning proof across the grades: A K–16 perspective* (pp. 153–170). New York, NY: Routledge.

Knuth, E., Zaslavsky, O., & Ellis, A. (2017). The role and use of examples in learning to prove. *Journal of Mathematical Behavior*, http://dx.doi.org/10.1016/j.jmathb.2017.06.002

Lannin, J., Ellis A., Elliot, R., & Zbiek, R. M. (2011). *Developing essential understanding of mathematical reasoning for teaching mathematics in prekindergarten-grade 8*. Reston, VA: National Council of Teachers of Mathematics.

Lesseig, K. (2016). Investigating mathematical knowledge for teaching proof in professional development. *International Journal of Research in Education and Science*, *2*(2), 253-270.

Lesseig, K., Hine, G., Na, G., & Boardman, K. (2019). Perceptions on proof and the teaching of proof: a comparison across preservice secondary teachers in Australia, USA and Korea. *Mathematics Education Research Journal*, *31*, 393–418.

Martin, W. & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education*, *20*(1), 41-51.

Melhuish, K., Thanheiser, E., & Guyot, L. (2020). Elementary school teachers' noticing of essential mathematical reasoning forms: justification and generalization. *Journal of Mathematics Teacher Education*, *23*, 35-67.

MES (2013). *Secondary mathematics syllabuses*. Ministry of Education Singapore: Retrieved from https://www.moe.gov.sg/education/syllabuses/sciences.

Na, G. (2014) Mathematics teachers' conceptions of proof and proof-instruction. *Communications of Mathematical Education*, *28*, 513-528.

Na, G. (2009). Teaching geometry proof with focus on the analysis-method. *Journal of Educational Research in Mathematics*, *19*, 185-206.

NCTM (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Peretz, D. (2006). Enhancing reasoning attitudes of prospective elementary school mathematics teachers. *Journal of Mathematics Teacher Education*, *9*, 381-400.

Simon, M. A., & Blume, G. W. (1996). Justification in the mathematics classroom: A study of prospective elementary teachers. *Journal of Mathematical Behavior*, *15*, 3-31.

Stylianides, A. J. (2016). *Proving in the elementary mathematics classroom*. Oxford University Press.

Stylianides, A. J., & Stylianides, G. J. (2006). Content knowledge for mathematics teaching: the case of reasoning end proving. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.), *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, *5*, 201-208. Prague: PME.

Stylianides, G. J. (2008). An analytic framework of reasoning-and-proving. *For the Learning of Mathematics*, *28*(1), 9–16.

Stylianides, G. J., & Silver, E. A. (2009). Reasoning-and-proving in school mathematics. In Stylianou, D. A., Blanton, M. L., & Knuth, J. (Eds.), *Teaching and Learning Proof Across the Grades: A K-16 Perspective* (pp. 235-249). New York, NY: Routledge.

Stylianides, G. J., Stylianides, A. J., & Philippou, G. N. (2007). Preservice teachers' knowledge of proof by mathematical induction. *Journal of Mathematics Teacher Education*, *10*(3),145-166.

Stylianides, G. J., Stylianides, A. J., & Shilling-Traina, L. N. (2013). Prospective teachers' challenges in teaching reasoning-and-proving. *International Journal of Science and Mathematics Education*, *11*, 1463–1490.

Yackel, E., & Hanna, G. (2003). Reasoning and proof. In J. Kilpatrick, W. G. Martin, & D. Shifter (Eds.), *A research companion to the Principles and standards for school mathematics* (pp. 227-236). Reston, VA: National Council of Teachers of Mathematics.