Journal of Educational Research in Mathematics
2020, Special Issue, 153 ~ 168

Middle-Grade Teachers' Reasoning with Fraction Division Tasks

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ABSTRACT

This study provides a fine-grained analysis of teachers' reasoning as they engage in quotitive fraction division tasks. In particular, this qualitative analysis considers the mathematical knowledge that six middle-grade teachers used in a professional development program and interviews. In all of the items, fraction division situations were approached through linear or area models. Analysis focused on three knowledge components with which teachers associated in reasoning fraction division tasks: partitioning operations, reasoning with quantitative units (referent units and levels of units), and interpretations of numerical expressions of division.

Key Words: Reasoning with quantitative units, Fraction division, Mathematical knowledge for teaching, Partitioning operations

BACKGROUND

The National Council of Teachers of Mathematics' standards for teaching and learning (NCTM, 2000) requires teachers to have a richer understanding of mathematics than traditionally required. The standards-based classrooms should create learning environments where students can engage in mathematics through various physical and visual representations and provide gateways to abstraction and generalization as their students develop the ability to mathematize problem situations (e.g., Dreyfus & Eisenberg, 1996). Despite that, a number of studies (e.g., Borko et al., 1992; Jaworski, 1994; Kazemi & Franke, 2004; Shifter, 1998) demonstrate that supporting teachers to meet the visions of mathematics reform is difficult. This is problematic given that those studies (e.g, Ball, Hill, & Bass, 2005; Charalambous, 2010; Cohen, 1990; Heaton, 1992; Lloyd & Wilson, 1998; Wilson, 1990; Hill, Blunk, et al., 2008) show the positive correlation between teachers' mathematical knowledge and the quality of their teaching practice. Teachers need to have a qualitatively different and significantly richer understanding of mathematics than most teachers currently possess in order to support students' learning.

Much of the research on teachers' knowledge in the past four decades have been framed in terms of knowledge constructs termed by Shulman (1986). These include subject-matter content knowledge, pedagogical knowledge, pedagogical content knowledge, and curricular knowledge. Many studies of mathematics teaching have investigated teacher knowledge under this frame (e.g., Ball, 1991; Ball, Lubienski, & Mewborn, 2001; Ball, Thames, & Phelps, 2008; Borko & Putnam, 1996; Ma, 1999). More recently, Ball and her colleagues have concentrated more on the unique mathematical knowledge teachers may have in their specific grade range – what some call specialized content knowledge. Specialized content knowledge is knowledge of mathematics that is used specifically in the work of teaching – for instance, knowledge that supports teachers' efforts to analyze students' novel approaches to computation and judge whether those approaches generalize to other examples. Together with basic grade-level content knowledge, Ball et al. (2008) termed these teaching-specific forms of knowledge *mathematical knowledge for teaching*. Growing attention to the perspective suggests that teachers need

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Please cite this article as: Lee, S. J. Middle-grade teachers' reasoning with fraction division tasks. *Journal of Educational Research in Mathematics* (2020). https://doi.org/10.29275/jerm.2020.08.sp.1.153

not only content knowledge that many educated adults have, but also knowledge specialized for teaching particular topics to students. Nonetheless, it is significant challenges for mathematics education research to identify knowledge that teachers need to support students' learning and assess that knowledge. The present study attempts to address the challenge in one area by studying middle grade teachers' knowledge of fraction division and the affordances and limitations of that knowledge in a sequence of division situations. The main research questions were (a) what were the primary partitioning operations and units with which teachers associated when they reasoned about fraction division problems? And (b) how did teachers modify or reorganize their knowledge components as they were faced with increasingly complex problem situations?

While researchers have provided considerable insight into students' conceptual understanding of rational numbers, the literature has yet to provide comparable insight into teachers' knowledge of these concepts. Specifically, in the present study I have investigated middle grade (Grade 5-7) teachers' capacities to reason with fractional quantities in quotitive division situations during a professional development program. Traditionally, division of fractions has often been taught by emphasizing the algorithmic procedure 'invert and multiply' without considering students' ways of constructing fraction division knowledge. While teachers do see the value of students' invented algorithms, they could not guide children to expand the construction because they did not understand the conceptual underpinnings of fraction division themselves. The topic is conceptually rich and difficult, as its meaning requires explanation through connections with other mathematical knowledge, various representations, and/or real-world contexts. This is the first step toward building a learning trajectory of teachers' ways of thinking, which can be extremely useful for thinking about how to build an effective professional development program and a teacher education program.

LITERATURE REVIEW

1. Teachers' Difficulties with Division of Fractions

Research on preservice and inservice teachers' understanding of fractions has demonstrated that, beyond knowledge of computation procedures, teachers' understanding of division with fraction is limited. A main result reported across studies is that teachers can confuse situations that call for dividing by a fraction with ones that call for dividing by a whole number or multiplying by a fraction (e.g., Armstrong & Bezuk, 1995; Ball, 1990; Borko et al., 1992; Ma, 1999, Simon, 1993). To illustrate, Ball (1990) asked 19 U.S elementary and secondary prospective teachers to model the expression $1\frac{3}{4} \div \frac{1}{2}$, and only 5 out of 19 teachers generated appropriate word problems. The teachers were likely to confound division by a half with division by 2 or multiplication by a half. Borko et al. (1992) documented that when Ms. Daniels, a prospective elementary teachers who performed student-teacher practice at the time, was asked by a child to explain why the invert-and-multiply algorithm worked, she attempted to explain by using an area model but failed to show fraction division; instead she modeled fraction multiplication, despite having completed a fair number of mathematics courses in her undergraduate program and being able to compute accurate answers with the invert-and-multiply method. The middle grades teachers that Armstrong and Bezuk (1995) investigated during a professional development program not only conflated division and multiplication situations when fractions were involved but also showed evidence of inflexibility with regard to the referent unit concept for fraction multiplication problems. Armstrong and Bezuk revealed that the teachers were extremely challenged to revisit multiplication and division of fractions. They claimed that their experiences with operating with fractions are symbolically oriented and algorithmic in nature.

Studies have also reported constraints of preservice teachers extending their knowledge of whole-number divisions to fractional contexts. Several studies were conducted to understand teachers' knowledge of fraction division using the primitive model framework of Fischbein, Deri, Nello, and Marino (1985). Tirosh and Graeber (1989) found that developing an understanding of the two models of division (partitive and quotitive) affected the choice of operations, multiplication or division. Simon (1993) revealed that inflexible and implicit use of the two models restricted prospective elementary teachers' success with fraction division problems. Behr et al. (1994) reported constraints of preservice teachers extending their knowledge of whole number division to fractional contexts. Harel and Behr (1995) interviewed 32 inservice teachers to identify and classify their strategies in solving rational number multiplication and division problems, looking for violations of basic intuitive models that were identified by Fischbein et al. (1985). It is not surprising that teachers had major difficulties in reasoning with division when they learned fractions without any conceptual linkage with whole numbers. To summarize, teachers confused situations calling for division by a faction with those calling for division by a whole number or multiplication by a fraction. Moreover, teachers showed a lack of capacity to devise word problems or to choose the word problems that appropriately represent the problem situations. Furthermore, as with children, teachers also appeared to be influenced by primitive models of division, which scholars blamed on teachers' whole number knowledge.

2. Teachers' Capacity to Reason with Quantitative Units

Studies that are presented in this section investigated teachers' knowledge at a finer grain size by emphasizing cognitive elements that are involved in context sensitive ways. They are different from the previous studies on teacher knowledge in that the former did not look at teachers' mathematical knowledge in broad categories such as common content knowledge, specialized content knowledge, etc. Particularly, the following studies are relevant to the current research in that they tried to explain where the constraints on teachers' knowledge of fraction division come from by analyzing teachers' capacities to reason with quantitative units, rather than simply documenting the constraints.

Izsák and his collaborators (Izsák, 2008; Izsák, Tillema, & Tunç-Pekkan, 2008) provided an account of middle school teachers' mathematical knowledge for teaching fraction multiplication and addition by focusing on their use of quantitative units. They found that teachers' abilities to make sense of students' reasoning differed based on the teacher's conceptual unit structure. The unit structures teachers employed shaped the purposes for which they used drawings when teaching fraction multiplication. To illustrate, a teacher who reasoned primarily with just two levels of units used a computed answer as a guide when developing an interpretation of her drawn representation, whereas a teacher who evidenced more consistent attention to three levels of units structures used drawings to infer a computation method (e.g., using the overlapping strategy in fraction multiplication context). Izsák (2008) argued that teachers would require more than an explicit attention to three levels of units structure in order to respond to students' thinking by "inferring students' understandings of the whole, parts of the whole, and parts of parts of the whole, and attending to the variety of ways that student might begin assembling three level unit structures as evidenced by their explanations and drawings" (p.107).

More recently, Izsák and colleagues used psychometric models to measure mathematical knowledge for teaching fraction arithmetic (Izsák, Jacobson, de Araujo, & Orrill, 2012; Izsák, Jacobson, & Bradshow, 2019). Teachers' performance to fraction arithmetic tasks for which they had to reason with drawn representations, was depended on their ability to reason with multiple levels of units (Izsák et al., 2012). The study extends Izsák's earlier research on measuring teacher knowledge for teaching fraction arithmetic not by methodologically including greater teacher populations but by including the focus of the present research, fraction division tasks. In Izsák et al. (2012), teachers who could reason with three level unit structures could reason with the tasks better than those who could only reason with two level unit structures. With recent advances afforded in psychometric modeling, Izsák, Jacobson, and Bradshaw (2019) surveyed over 990 teachers and argued teachers' attention to referent units as one of the five knowledge resources to reason with aforementioned fraction tasks. In addition to their attention to referent units, partitioning and iterating, appropriateness, and reversibility were other four components of knowledge. The result corroborated the result of S. J. Lee, Brown, and Orrill (2011) by underscoring teachers' ability to attend to referent units in solving fraction arithmetic problems involving drawn representations. In S. J. Lee et al. (2011), attention to referent units was not enough to support teachers' sophisticated problem solving strategies. Having flexibility with such units was imperative for them to reason with quantitative units. Such observation was not only limited to inservice teachers but also to preservice teachers in United States (e.g., S. J. Lee, 2011, 2012; M. Y. Lee, 2017) and inservice teachers in Korea (e.g., Kim, Kim, & S. J. Lee, 2016). Teachers' ability to reason with three levels unit-structure is a precursor to their flexibility with referent units.

While several studies emphasized the role of reasoning with multi-level unit structures in developing multiplicative reasoning, most of the research on this area has concentrated on students (e.g., Izsák, 2005; S. J. Lee & Shin, 2015, 2020; Shin, S. J. Lee, & Steffe, 2020; Smith, 1995; Steffe, 1988, 1992, 1994, 2001, 2003, 2004; Steffe & Olive, 2010). Only a few researchers (Behr, Khoury, Harel, Post, & Lesh, 1997; Izsák, 2008; Izsál et al., 2008) have extended this area by studying teachers' capacities to reason with multi-levels of unit-structures when reasoning about situations that call for division of fractions. The current study is contributing to the field of teacher knowledge research by investigating teachers' knowledge of fraction division in a finer grain size and delineating knowledge components that seem essential for developing teachers' specialized knowledge for teaching.

THEORETICAL CONSTRUCTS

In this section, I provide several theoretical constructs closely related to problem solving behaviors and mental operations involving quotitive fraction division problems. These constructs played a crucial role for analyzing the collected data of the study, and thus would be essential for better understanding of the study. The meaning of *referent unit* adopted in this study is similar to the term *adjectival quantities* that Schwartz (1988) defined. According to Schwartz, "All quantities that arise in the course of counting or measuring or in the subsequent computation with counted and/or measured quantities have referents and will be referred to as adjectival quantities." (p. 41) He further stated that all quantities have referents and that the "composing of two mathematical quantities to yield a third derived quantity can take either of two forms, referent preserving composition or referent transforming composition." (p. 41). In other words, the referent unit of the third quantity remains the same in addition and subtraction, but it is different from either of the two original quantities in multiplication and division.

The fundamental operation in teachers' measurement division situations, as reported in detail later at the 'Result' section, was a *unit-segmenting operation*. It originates from the Fractions Project (see Steffe & Olive, 2010 for comprehensive work of the Fractions Project), which studied children' construction of fractions. The unit-segmenting operation entails the operation of segmenting the dividend by the divisor. To find the quotient of the simple measurement fraction division situation using drawn representations, say $2 \div \frac{1}{5}$, one can use length quantity starting with a two-level units structure of 2 as in *Figure 1a*. This requires attention to just two levels of units, the whole and the two units of one. To determine how many groups of one-fifth are in two, one needs to use a unit-segmenting operation and segment the unit of two by the measurement unit of one-fifth.

One may construct the relevant quantity by partitioning one part into five pieces (reasoning with two levels of units 1/5 and 1) and another part into five pieces (reasoning with two levels of units 1/5 and 1 a second time), and iterate the smallest unit ten times until it exhausts the original unit (*Figure 1b*). This involves reasoning with two levels of units a third time (a fifth and a unit containing ten fifths). Alternatively, one might recursively partition by subdividing each of two ones into five pieces (*Figure 1c). Recursive partitioning operation* is a fundamental operation for students' (Steffe, 2004; 2005) and teachers' fraction multiplication knowledge (Izsák, 2008). Steffe defined it as taking a partition of a partition in the service of a non-partitioning goal. In contrast to the first solution, recursive partitioning involves applying distribution of a partitioning operation across the parts of another partition (Steffe & Olive, 2010; Shin & S. J., Lee, 2018; Shin, S. J. Lee, & Steffe, 2020). In other words, it involves reasoning with three levels of units in which five of the smallest units are nested within each mid-level unit (two units of one), and so there must be 2 x 5 = 10 units in the whole unit.

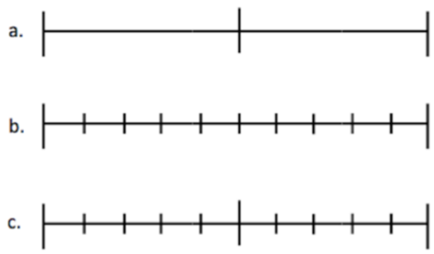


Figure 1. Determining the quotient of $2 \div \frac{1}{5}$ a. Constructing a two-level units structure for 2. b. Using iteration and two levels of units. c. Using recursive partitioning and three levels of units.

Common partitioning operation refers to the partitioning operation that one uses to coordinate and compare two iterable composite units until a common number to be used in partitioning is found (Shin & S.

J., Lee, 2018). It requires units-coordination at three levels of units–that is, a coordination of two sequences of composite units. Using common partitioning operations, one could find the commensurate fractions for the dividend and the divisor quantities using the co-measurement unit. A *co-measurement unit* is defined as a measurement unit for commensurable segments, that is, segments that can be divided by a common unit without remainder (Olive, 1999). For example, in $\frac{2}{3} \div \frac{1}{7}$, one can start with a two-level units structure of $\frac{2}{3}$ and $\frac{1}{7}$ as in *Figure 2a* and *2b*. If one tries to use the two two-level units structures of $\frac{2}{3}$ and $\frac{1}{7}$, it is very complicated to find how many groups of $\frac{1}{7}$ are in $\frac{2}{3}$ as in *Figure 2c*.

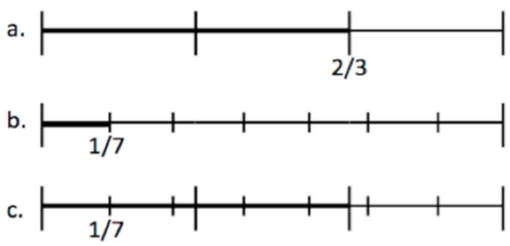


Figure 2. Determining $\frac{2}{3} \div \frac{1}{7}$,. a. A two-level units structure for $\frac{2}{3}$. b. A two-level units structure for $\frac{1}{7}$. c. Using two of two-level units structures of $\frac{2}{3}$ and $\frac{1}{7}$.

Using common partitioning operations, one can find two-thirds of a unit stick using one-seventh of the stick by finding $\frac{1}{21}$ as a co-measurement unit for both one-third and one-seventh. Using the co-measurement unit as a base, one could find *commensurate fractions* for two-thirds and one-seventh as fourteen twenty-first and three twenty-first as in *Figures 3a* and *3b*.

Finding commensurate fractions entails reasoning with three-level units structures (Steffe & Olive, 2010). The term is chosen to describe a case when one uses drawings of quantities to figure out, in conventional terms, equivalent fractions. By the common partitioning operation, one could attend to two three–level units structures in determining how much of $\frac{3}{21}$ is $\frac{14}{21}$ and thus to determine the quotient for $\frac{2}{3} \div \frac{1}{7}$, as in *Figure 3c*.

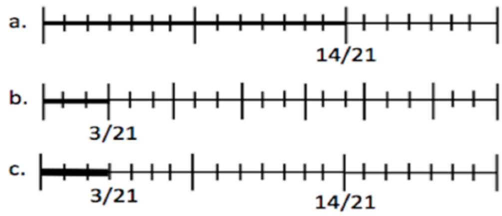


Figure 3. Determining the quotient of $\frac{2}{3} \div \frac{1}{7}$. a. A three-level units structure of $\frac{2}{3}$. b. A three-level units structure of $\frac{1}{7}$. c. Two three-level units structures of $\frac{2}{3}$ and $\frac{1}{7}$.

METHODOLOGY

The present study was conducted within the activities of the project, *Does it Work?: Building Methods for Understanding Effects of Professional Development* (DiW), funded by the National Science Foundation (NSF). The data came from a 40-hour professional development course on fractions, decimals, and proportions. Thirteen mathematics teachers came from a large urban district in the Southeast part of the United States. Of the teachers, one taught in grade 5, and the others taught in grade 6 or grade 7. They were recruited through their district office and were offered a small stipend. The course was designed to prepare teachers for new curriculum standards in their state, which were consistent with many of the NCTM (2000) standards. It requires teachers not only to compute efficiently and accurately with fractions, decimals, and proportions, but also to reason about them embedded in problem situations. Many of the problems created opportunities for teachers to generate and interpret drawn models for multiplication and division of fractions and decimals with length or area units. Teachers constructed drawn models using either paper and pencil or, software called Fraction Bars (Orrill, 2003). The software opened the possibility for the participating teachers to create and enact various operations (i.e., partitioning, disembedding, iterating, pullout, break, etc.) on various geometric figures such as rectangles and squares.

The DiW project administered a multiple-choice pre-assessment, post- assessment, and delayed-post-assessment that was developed by modifying (with permission) the middle grades Learning Mathematics for Teaching measure of the Mathematical Knowledge for Teaching project (Hill, 2007). In addition, follow-up interviews were conducted with eight teachers after each assessment. All meetings and interviews were videotaped using two cameras, and these two sources were then mixed to create a *restored view* of the event (Hall, 2000). As a research staff of the project, the author participated in all the class sessions, interviews, and administrating the assessments. The data collected for this qualitative study included videotaped lessons, reflections, and lesson graphs of the 5 relevant instructional meetings, and pre-, post-, and post-post assessment interviews and interview graphs for eight of the 14 teachers.

Data analysis occurred in stages. The first stage was ongoing analysis throughout the implementation of the professional development course. The DiW principal investigator, the facilitator[1] of a course, and the author debriefed at the end of each session. Discussion focused on how the participating teachers were making sense of the content, which became basic resources in planning for future sessions. Immediately after each session, the facilitator created annotated timelines of each session using a lesson graph format. These summaries provided a written description of teachers' mathematical activities and interactions with the instructor. From the lesson graph, I wrote down emerging key points in teachers' reasoning that were taken into account for the next section. Teachers' partitioning operations and flexibilities with units were the two key knowledge elements that emerged at the time. The timeline and descriptions were highlighted whenever observing the instances where teachers used the two knowledge components.

The second stage of analysis was retrospective analysis. From the initial lesson graphs that outlined the key incidents of the class meetings, key episodes were identified and developed second lesson graphs that only contained the episodes related to the research focus. Four sequences emerged through the retrospective analysis and I used the four sequences as the coding schemes to answer the second research question. During the coding process of using the four sequences, a hypothesis was generated that teachers seemed to use more sophisticated operations as the sequence increased. Then, I went back and forth between the video data and lesson graphs several times and found that teachers' initial conceptions of unit-segmenting operations were reorganized in each sequence by using more sophisticated partitioning operations and by attending to the referent unit. In the third round of analysis, all eight teachers' assessment interview data were analyzed. From the restored views, I made interview graphs, which had the same format as the classroom lesson graphs. The interview graphs were produced to describe how the interviews proceeded along with snapshots of teachers' works. A timeline and brief analytical notes were also included.

RESULTS

The participating teachers revealed distinguishable characteristics of their measurement fraction division knowledge across four different types of division situations: 1) When the divisor quantity measures out the dividend quantity evenly (Sequence 1); 2) When the divisor quantity does not measure out the dividend quantity evenly (Sequence 2); 3) When denominators of the dividend and the divisor quantities are relatively prime (Sequence 3); 4) When the divisor quantity is larger than the dividend quantity (Sequence 4). Within each of these situations, a sequence of tasks, in which the divisor and dividend were combinations of various whole numbers and fractions, were posed. The unit-segmenting operation, the operation of segmenting the dividend by the divisor, was a fundamental operation for the teachers' solving measurement division problems over the whole sessions, but their capacities to deal with the problems in each sequence were differed in terms of not only partitioning operations but also levels of units that were available to them. *Table 1* summarizes the operations that teachers had used or might have used in each sequence.

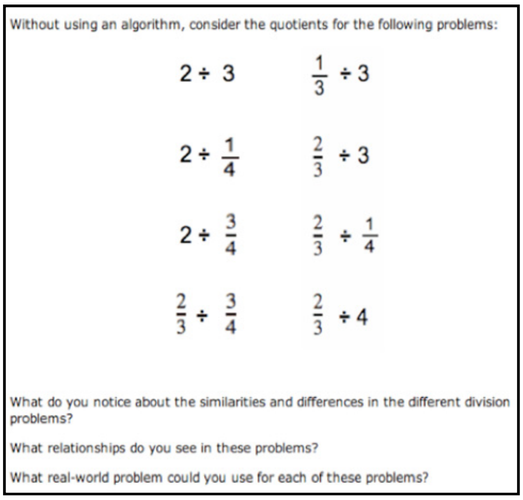
While a few teachers had been using a measurement interpretation to division (i.e., quotitive division) as evidenced by the pre-assessment interview, the class as a whole did not talk about it until the discussion of Task 1 in *Figure 4* below. How teachers interpreted the numerical expressions of division framed their choices of division models (partitive or quotitive). In general, except for the cases when the divisor was a whole

1 The role of the facilitator in the professional development was to ask teachers to provide justification for claims, question incomplete or incorrect mathematical ideas, provide counterexamples, and push for generalizations. She refrained from providing answers to the task teachers solved during the course as a way to de-emphasize her role as an authority figure. Please read Brown(2009) for more description of her role as a researcher - facilitator.

number, teachers used unit-segmenting operations and used an interpretation of either ‘How many times one

Table 1. Summary of operations for each sequence of fraction division

	Whole number dividend	Fraction dividend
Sequence 1	Unit-segmenting operation using two levels of units; Or, recursive partitioning operation for unit-segmenting operation	
Sequence 2	Unit-segmenting operation using two levels of units; Or, recursive partitioning operation for unit-segmenting operation	Unit-segmenting operation using two levels of units; Or, common partitioning operations (with a cross partitioning strategy) for unit-segmenting operation (co-measurement unit and commensurate fractions)
	Establishing a part-whole relationship between the leftover quantity and the divisor quantity (novel situation) was a key knowledge.	
Sequence 3	Not applicable	Sequence 2 Sequence 4
Sequence 4	Establishing a part-whole relationship between the dividend and the divisor quantity using two levels of units; Or, recursive partitioning operation for establishing a part-whole relationship between the dividend quantity and the divisor quantity.	Establishing a part-whole relationship between the dividend and the divisor quantity using two levels of units; Or, common partitioning operations (with a cross partitioning strategy) for establishing a part-whole relationship between the dividend quantity and the divisor quantity.


Without using an algorithm, consider the quotients for the following problems:

$2 \div 3$ $\quad \frac{1}{3} \div 3$

$2 \div \frac{1}{4}$ $\quad \frac{2}{3} \div 3$

$2 \div \frac{3}{4}$ $\quad \frac{2}{3} \div \frac{1}{4}$

$\frac{2}{3} \div \frac{3}{4}$ $\quad \frac{2}{3} \div 4$

What do you notice about the similarities and differences in the different division problems?

What relationships do you see in these problems?

What real-world problem could you use for each of these problems?

Figure 4. Exploring division with fractions: Task 1.

can subtract the divisor from the dividend (repeated subtraction)’ or ‘How many groups of the divisor quantity fit in its dividend.’ Even though teachers accepted both as valid quotitive approaches to the division, there was a difference in the teachers’ abilities to reason in the sequence of division situations using drawn representations and to choose word problems that depicted the measurement division. For instance, Donna said she did not accept the word problem that modeled the measurement division situation because the divisor quantity was bigger than the dividend quantity; hence, a repeated subtraction strategy could not be used.

Sequence 1: When the Divisor Partitions the Dividend Evenly

This was the case in which the divisor quantity evenly measured out the dividend quantity (e.g., $2 \div \frac{1}{4}$). When the dividend quantity was a whole number in Sequence 1, teachers used either recursive partitioning operations for unit-segmenting operations based upon reasoning with three levels of units or unit-segmenting operations with two levels of units. To illustrate, Claire[2] knew the quotient of $2 \div \frac{1}{4}$ was eight before she even showed us her model. When she was asked to model the problem using a drawn representation, she immediately drew a box partitioned into two by four parts and stated that the quotient was eight because there were eight groups of one-fourth in two. Her immediate response to the problem likewise suggests that she used the interpretation, ‘How many groups of one-fourth are in 2?’ and generated a three level unit structure in which the length of 2 was the largest unit, the length between 0 to 1 was the mid-level unit, and the length of $\frac{1}{4}$ was the smallest unit. In order words, she conceived the whole 2-part bar as one unit of two, two units of one, and eight units of one fourth. On the other hand, Carrie used a unit-segmenting operation

2 All names used are pseudonyms.

with two levels of units for the same problem, she constructed one bar and split it into four parts and know that there are four-fourths in one. She then knew that there were another four-fourths in one by re-presenting a bar that was split into four parts.

In the case of fraction dividend in Sequence 1, similar to the situation of whole number dividend, teachers could[3] use unit-segmenting operations with two levels of units, or they could activate recursive partitioning operations for their unit-segmenting operations. In order to measure $\frac{1}{3}$ in terms of $\frac{1}{9}$, one can draw another same size whole bar below the first bar and then partition it into nine parts (i.e., two levels of units) as in *Figure 5*. Because the two bars were lined up so that one could clearly see that three ninths in the second bar fit in the third in the first bar, he may say the quotient for $\frac{1}{3} \div \frac{1}{9}$ is 3.

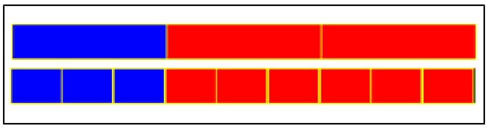


Figure 5. Using a unit-segmenting operation with two levels of units of $\frac{1}{3}$ and $\frac{1}{9}$.

In contrast, when one uses three levels of units to find the quotient, one does not need to line up the two bars to find out how many divisors fit in its dividend but could start with the third bar and split it into three units knowing one ninth is one third of one third; hence there are three ninths in one third. In brief, a person's ability to take the result of recursive partitioning for granted in that knowing one ninth as one third of one third, requires the person's attention to the three levels of units, that is, to view a whole as a unit of one, three units of thirds, and nine units of one ninth.

Sequence 2: When the Divisor Does Not Partition the Dividend Evenly

When the divisor quantity did not partition the dividend quantity evenly, it was not enough for teachers to use unit-segmenting operations. They also needed to establish a part-to-whole relationship between the leftover quantity and the divisor quantity.

3 'Could' is used to underscore that analysis of this case is purely based on my hypothesis. Such type was not asked to teachers during the PD session.

As soon as Claire finished explaining her method for solving $2 \div \frac{1}{4}$ using area model, she started to show how she would solve $2 \div \frac{3}{4}$ using the same area model.

Protocol 1: Claire's limited attention to the referent unit of the leftovers.

CL[Claire]: $2 \div \frac{3}{4}$ is three-fourths *[X in Figure 6b]*, three-fourths *[Y in 6b]*, two and (she writes the number 2 in the paper as in *Figure 6a*), I have 2 and $\frac{1}{2}$ (she writes $\frac{2}{4}$ first then changes it into $\frac{1}{2}$ as in *Figure 6a*).

FAC[Facilitator]: So that one $[2 \div \frac{3}{4}]$ is two groups of three-fourths and then you have two leftovers (She points at the two leftover pieces)?

CL: Two-fourths of [Pause] Oh! The two *[z in Figure 6b]* would be two-thirds because yeah [it takes] three to make a group, wouldn't it?

FAC: Hang on. So we went from two-fourths to two-thirds?

CL: Oh, well, it is two-fourths I was right the first time, wasn't I? Those *[z in Figure 6b]* are fourths.

FAC: (Laughter) Hang on. Hang on. You are going too quickly. So two fourths or two thirds, let's think about it (She tries to replace the $\frac{1}{2}$ with $\frac{2}{3}$ as in *Figure 6a*). So you are saying these *[z in 6b]* are fourths, so there are two [one-fourth] of them. Now when we talk about two-fourths, we are talking about what referent unit? When you say a fourth (halted).

CL: The fourth is of one whole, but new *[referent]* units *[that I use]* are three-fourths [X and Y in *Figure 6b*], so they (she swiftly moves two pieces in z of *Figure 6b*) would be two thirds.

FAC: So the referent unit we really want is the three-fourths.

CL: Three-fourths is one whole.

FAC: And you said you have two out of three (points at three that Claire circled).

CL: Yeah, three.

FAC: Does that make sense [Rose]?

CL: Two and Two-thirds.

FAC: So when you think of division, Claire, you are asking "How many of these [three fourths] would fit that [2]?

CL: Right, that is what I am asking.

FAC: So how many groups of these [three-fourths] would fit that [two]?
CL: Yes.

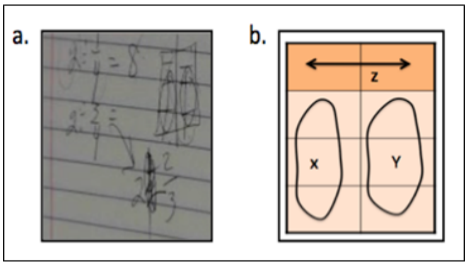


Figure 6. a. Claire's actual writing and drawing for $2 \div \frac{1}{4}$ and $2 \div \frac{3}{4}$. b. Reconstruction of Claire's model of $2 \div \frac{3}{4}$.

Note that Claire used a recursive partitioning operation to use a unit-segmenting operation to show $2 \div \frac{1}{4}$ when she used a length quantity in Sequence 1. Her recursive partitioning operation was an assimilating operation of her unit-segmenting scheme, and it allowed her to find the quotient 8 immediately without having to segment the unit 2 by $\frac{1}{4}$. On the other hand, in order to find the quotient for $2 \div \frac{3}{4}$, she needed to actually segment the bar as described in Protocol 1. The fact that she knew she could use the same bar, which was a product of her recursive partitioning operation, suggests that she was attending to the unit of two at three levels: one unit of two, two units of one, and eight units of one-fourth. Using the unit-segmenting operation, she counted the number of three-fourths by circling (X and Y in *Figure 6b*.) the three pieces from each column and wrote down $2\frac{2}{4}$ on the paper. The result of activating the unit-segmenting operation was two and two-fourths, but there was no perturbation for encountering the unit that could not be clearly segmented by the segmenting unit. It displays that she conflated the unit to be used in segmenting (i.e., a measurement unit) with the mid-level unit of the three levels of units-structure (the unit of one) to which she was attending by evoking a recursive partitioning operation, and determined the leftover quantity as two-fourths.

Facilitator noticed Claire's conflation of referent units in measuring the leftover quantity and asked her, "So that one [$2 \div \frac{3}{4}$] is two groups of three-fourths and then you have leftovers?" by pointing at the leftover pieces on her picture. Claire saw three pieces and the two pieces on her model and established the part-to-whole relationship by visually comparing the unit of two and the unit of three and changed her answer to $2\frac{2}{3}$. She realized that it took three of the smallest pieces to make a group by looking at her drawing and found that the two leftover pieces would be two-thirds because it took "Three to make a group." Nevertheless, she switched the answer back to $2\frac{2}{4}$ because the fact that each piece was one-fourth seemed also prominent in her thinking at the time. Given that Claire had two points of view on one situation, it is not surprising that she was a bit unstable. Even though Claire established the part-whole relationship by visually comparing the leftover two pieces with the measurement unit in her drawing, Claire conflated the units again when she reflected that each piece was one-fourth. It seems no surprise for Claire to a priori conceive the smallest piece as one-fourth knowing that she used a recursive partitioning operation to solve $2 \div \frac{1}{4}$ pointing at each of the eight pieces in her two-by-four area model. Finally, Claire corrected her answer when the facilitator asked her what the referent unit was if the leftover two pieces were 'two-fourths'. Claire realized that she was supposed to use the measurement unit [three-fourths] as the referent unit to the leftover quantity. Using the measurement unit as the referent unit to the leftover quantity, she established a part-to-whole relationship using a visual support from her drawing.

Sequence 3: When Denominators of the Dividend and the Divisor are Relatively Prime

When denominators of the dividend and the divisor were relatively prime, common partitioning operations for unit-segmenting operations were provoked. Some teachers used recursive partitioning operations while others using a common denominator strategy in the situation of conducting common partitioning operations. By constructing co-measurement units through common partitioning operations, the teachers could find commensurate fractions for the dividend and the divisor quantity. When teachers faced the problems in Sequence 3, some teachers needed to use either a common partitioning or a cross partitioning operation. Most of those teachers used the term 'common denominator' whenever they had to find a common partition to transform two fractions into the fractions with like denominators. At first it seems the teachers were adopting an algorithm for fraction division, but later I found that the process of finding a common denominator was based on their necessity for making a commensurate fraction. In other words, the teachers' use of the common denominator method was more than procedural. They did not use it to calculate the numerical answers to the given division problems but to clearly show commensurate fractions for the dividend and the divisor quantities. Thus, their common denominator method was associated with common partitioning operations. In the following, I illustrate teachers' common partitioning operations by providing the number line model of Claire (*Figure 7b*).

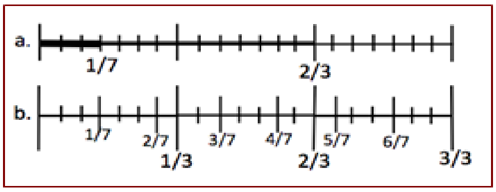


Figure 7. a. A typical representation of common partitioning operation for $\frac{2}{3} \div \frac{1}{7}$. b. Reconstruction of Claire's common partitioning operation for $\frac{2}{3} \div \frac{1}{7}$.

When Claire was asked to justify the number line model for $\frac{2}{3} \div \frac{1}{7} = \frac{14}{3}$ that she had already drawn (like a. in *Figure 7*) during the post-assessment interview, Claire started with a number line and partitioned it into seven parts and labeled each seventh. Then she recursively partitioned each seventh into three parts and produced units of twenty-first. Finally, she drew over each third with the vertically longer line exactly like in *Figure 7b*. She was even aware that she had produced a 'common partition' between the dividend and the divisor quantities. As usual, Claire reminded herself of the question 'How many one-sevenths are contained in two-thirds?' and found commensurate fractions for $\frac{2}{3}$ and for $\frac{1}{7}$ using a co-measurement unit, $\frac{1}{21}$. Claire knew that a commensurate fraction for $\frac{2}{3}$ was $\frac{14}{21}$ and for $\frac{1}{7}$ was $\frac{3}{21}$ from her number line model. When she determined the quotient, she knew four one-sevenths fit clearly in the two thirds, and the two pieces were leftovers. Labeling each seventh and third and drawing vertically different lengths of sevenths and thirds helped her measure the size of the two leftover pieces as two out of three. Claire's model might not be the case of a typical common partitioning operation for $\frac{2}{3} \div \frac{1}{7}$ (like in *Figure 7a*) in that she started with a 7-part bar instead of a 3-part. I, however, attribute her to activation of a common partitioning operation because she coordinated two three levels of units of fractions $\frac{2}{3}$ and $\frac{1}{7}$ and produced a co- measurement unit of $\frac{1}{21}$.

In contrast that Claire explained $\frac{2}{3} \div \frac{1}{7}$ using a number line model in post assessment interview (*Figure 7*), she cross partitioned an area model by partitioning a bar vertically and horizontally and coordinated two composite units of the divisor and the dividend in delayed post assessment interview (*Figure 8*).

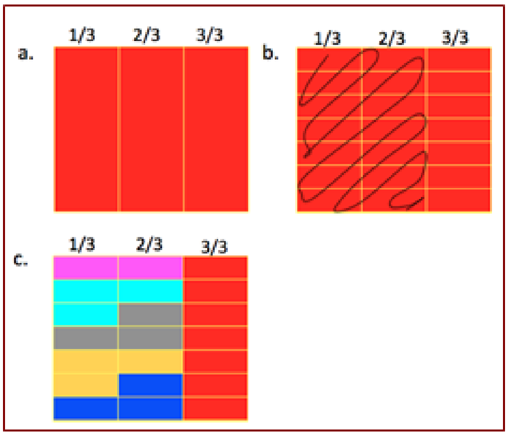


Figure 8. Claire's area model to demonstrate $\frac{2}{3} \div \frac{1}{7}$ in order of a., b., c.

Her common partitioning operation using a cross-partitioning strategy was different from that in the former as it provided her with a simultaneous repartitioning of each part of an existing partition without having to insert a partition into each of the individual parts. To illustrate, when the interviewer asked Claire to explain how she would like to model $\frac{2}{3} \div \frac{1}{7}$, she drew an area model like in *Figure 8a* and vertically split a bar into three parts and labeled each column $\frac{1}{3}$, $\frac{2}{3}$ and $\frac{3}{3}$. Then she horizontally split the 3-part bar into seven parts and shaded in two columns to indicate the dividend two-thirds like in Figure 8b. Then she reflected on the problem and asked, 'How many sevenths fit in two-thirds?' which was the interpretation of fraction division that she usually used. Then she continued to say,

> One, two, three (she shades three blue pieces in *Figure 8c*), so three of those little sections is the seventh, one, two, three (counted yellow

pieces in c.) two [sevenths], one, two, three (counted gray pieces in c.), three [sevenths] and one, two, three, four [sevenths] (counted sky blue pieces in c.), so it would be four and two thirds (wrote down $4\frac{2}{3}$ besides the model).

After she counted four units of three from the area model she drew like in Figure 8c, she used the unit of three to measure the two leftover pieces and correctly stated the quotient for $\frac{2}{3} \div \frac{1}{7}$ as $4\frac{2}{3}$. Using a cross partitioning strategy allowed her to find the quotient to the division problem without explicitly using commensurate fractions. Note that she said one-seventh was equal to three pieces as she used the commensurate fractions of $\frac{3}{21}$ and $\frac{14}{21}$ in her number line model. Moreover, she iterated three pieces to find how many groups of three in 14. With the result of a cross partitioning strategy[4], she could use whole-number three-level units structure (i.e., the length of one, the length of three, and the length of 14). As a matter of fact, in the class, she used a cross partitioning strategy to show the whole class how she found the quotient for $\frac{2}{3} \div \frac{3}{4}$ and she knew $\frac{2}{3}$ was $\frac{8}{12}$ and $\frac{3}{4}$ was $\frac{9}{12}$. But, again, she did not need to explicitly attend to the co-measurement unit $\frac{1}{12}$ to determine the quotient for $\frac{2}{3} \div \frac{3}{4}$. Whole-number three levels of units were enough: the length of one, the length of eighth, and the length of nine.

Sequence 4: When Divisor Is Bigger Than the Dividend

When the divisor quantity was bigger than the dividend quantity, teachers needed to find an alternative way to measure the dividend quantity with the divisor quantity. They could not iterate the divisor or count the number of divisors contained in the dividend to find the quotient because the divisor quantity was bigger than the dividend quantity. Teachers determined the size of the dividend quantity by setting up the part-to-whole relationship between the two quantities (e.g., $\frac{8}{12} = \frac{8}{9}$ of $\frac{9}{12}$ for $\frac{2}{3} \div \frac{3}{4}$). The activation of the part-to-whole relationship was not new. In Sequence 2, when the divisor quantity did not evenly measure out the dividend quantity, some teachers used it to deal with leftovers. For example, to determine a quotient for $2 \div \frac{3}{4}$ (Sequence 2) without using the algorithm, Claire generated a 2-part bar and recursively partitioned each part into four parts. After she counted three-fourths from each part, she attended to the part-whole relationship in which the leftover quantity was the part and the measurement unit, three-fourths was the whole and found that two-fourths was two-thirds of three-fourths. In other words, when teachers assimilated a problem situation in Sequence 4 to a subdomain of Sequence 2, they were able to solve the problem in no conflict with their current meaning of division, that is, 'How many of the divisor quantities are contained in the dividend quantity'. While Keith, Claire, and Donna established the part-whole relationship between the dividend and the divisor quantities to determine a quotient for $\frac{2}{3} \div \frac{3}{4}$, none of them seemed to recognize that it was a similar situation to the one where they had to deal with the leftovers in Sequence 2. As a matter of fact, the teachers thought that the situation was very different from others like $2 \div \frac{1}{4}$ (Sequence 1) or $2 \div \frac{3}{4}$ (Sequence 2) because they could not use a unit-segmenting operation.

When a divisor was less than a dividend (e.g., Sequence 1 and 2), teachers generally started with the dividend quantity for whichever models (number line or area) they used. However, when a divisor is larger than a dividend (Sequence 4), a couple of teachers (who could model the problem in Sequence 4 using length or area quantities) started by showing the divisor quantity. Only one teacher, Keith not only started with the divisor quantity but also in its commensurate fraction form using the co-measurement unit of one-third and one-fourth. Thus, Keith did use a common partitioning operation by coordinating the two iterable composite units of three and four. Later in the class, Keith stated he chose nine-twelfths among all equivalent fractions for three-fourths because one-twelfth was the co-measurement unit of both nine-twelfths and eight-twelfths.

DISCUSSION

Improving teachers' knowledge of mathematics is crucial for improving the quality of instruction (Ball, Lubienski, & Mewborn, 2001; Ma, 1999; Mewborn,

4 Although the strategy aided the teachers to reason with fractional quantities, there is a drawback to introducing the strategy too early to students for their learning of fraction arithmetic concepts. The drawback is discussed in Shin & S. J. Lee (2018).

2003; Sherin, 1996), and efforts to improve the quality of classroom instruction have led to increase attention to promoting the development of teachers' mathematical knowledge for teaching. Especially, research on teachers' understanding of fraction division (e.g., Ball, 1990; Borko, 1992; Simon, 1993; Ma, 1999) has demonstrated that one or more pieces of an ideal knowledge package (Ma, 1999) for fraction division is missing. Although previous studies (e.g., Ball, 1990; Borko, 1992; Simon, 1993; Ma, 1999; Tirosh & Graeber, 1989) have stressed errors and constraints on teachers' knowledge of fraction division, few studies have been conducted to explore teachers' knowledge of fraction division at a fine-grained level (Izsák, 2008). The present study adds to the small but growing number of studies reporting that how teachers reason with models for fraction arithmetic depends on their ability to coordinate fine-grained knowledge components such as partitioning operations, flexibility with referent units, and coordination of multi-level unit structures(e.g., Izsák, Jacobson, de Araujo, & Orrill, 2012; Izsák, Jacobson, & Bradshaw, 2019).

A few teachers like Claire could reorganize their fraction division knowledge by refining their partitioning operations and units through the sequence of problem situations in which the mathematical relationship between the dividend and the divisor became increasingly complex. While unit-segmenting operations were key operations when the divisor was smaller than the dividend, establishing part-whole reasoning between the dividend and the divisor was also a key operation when the divisor was bigger than the dividend. Moreover, common or cross-partitioning operations emerged as the teachers engaged in the division problems where the denominators of two fractions were relatively prime. Despite that few teachers were explicitly aware of it, from a researcher's point of view, the teachers seemed to have previously established a part-whole relationship between the two quantities when they needed to quantify a leftover quantity in terms of a measurement unit. The teachers established this quantitative relationship between the two quantities by using their unit-segmenting operations.

Furthermore, the teachers did not want to think about the situations that they could not easily conceptualize with their existing conceptions of divisions. In general, those teachers preferred to use a quotitive division model when fractions were involved as divisors and a partitive division model when whole numbers were involved as divisors. Their preference to a particular division model characterized their ability to devise valid word problems for both division models. Regardless of the fact that the question explicitly asked the teachers to come up with both models, only 3 out of 14 teachers came up with reasonable word problems and others did not offer a partitive word problem when a divisor was a fraction.

The teachers' coordination of two three-levels-of-units structures activated more sophisticated partitioning operations and supported the teachers making sense of more complex fraction division situations. On the other hand, the teachers, who used two levels of units across each measurement fraction division sequence, did not demonstrate more sophisticated partitioning operations (i.e., recursive partitioning operations and common partitioning operations) to activate unit-segmenting operations. In addition, when the teachers established part-whole reasoning between the two quantities, their ability to use the measurement unit as a referent unit was critical, and it was impossible without the teachers' coordination of two three-levels-of-units structures.

The present study suggests that the knowledge components found in the previous research literature about children's fractional knowledge appeared in the participating teachers' mathematical activities with fraction problems and further turned out to be essential for their mathematical thinking in the context of division problems. Although there are some compatibilities between children's and teachers' ways of knowing so that applying the results from research with children could be a viable way to start, I began to realize that the development of teachers' knowledge should differ from that observed in children because the teachers were already well equipped with procedural knowledge. To elaborate, some participating teachers' common partitioning operations were evoked by their strategy of finding a common denominator between two fractions. From researchers' point of view, the teachers brought forth common partitioning operations by themselves, but they seemed to believe they utilized an algorithm. They were referring to the algorithm in finding a common denominator for two fractions, which was a procedural strategy that they usually used in fraction addition or subtraction problems. Such unawareness might be the result of their revisiting the content over and over in their classroom. One may think that it has no harm as long as the teachers' algorithmic, procedural knowledge was associated with the (mental) operations. However, I suspect that this may cause serious problems when the teachers go back to their classrooms. Just taking a common partitioning operation as an algorithm or a strategy ignores assimilating structures (recognition templates), conducted partitioning operations, and derived (expected) results of the mental operation, which would lead their students to accept it as one of the problem-solving skills. Thus, teachers need to be explicitly aware of the associations they make between the operations and the procedural algorithms.

IMPLICATION AND FUTURE RESEARCH

Although past research has stressed errors in and constraints on teachers' knowledge of partitive and quotitive fraction division, relatively few studies have been devoted to investigations of teachers' knowledge in terms of their mathematical operations. Especially little has been done in examining teachers' reasoning about fractional quantities in terms of conceptual units. Teachers' abilities to conceptualize quantities involved in word problems or in drawn representations into the abstract units turned out to be one of the important knowledge components. This suggests that teacher education activities should place deliberate emphasis on quantitative reasoning, interpreting and writing word problems, and creating and interpreting drawn representations so that they can develop flexibilities with quantitative units (e.g., units of area/length).

Future research should explore teachers' knowledge within and beyond the measurement division sequences that I have identified in this study. Even though I could not provide confirmable analysis concerning a plausible connection between sequence 4 and sequence 2 during the professional development course , the teachers, who could attend to measurement units using their common and cross partitioning operations in sequence 2, seemed to deal with the sequence 4 problems with little perturbation as opposed to those who used unit-segmenting operations with two levels of units in sequence 2. Because only one problem from the data fell in sequence 4, it is hard to establish this conjecture in general. Thus, this conjecture is worthy of further study.

Future research should also consider teachers' knowledge of fraction division in their classroom setting by coordinating the teachers' knowledge with their students' one. A similar study has already been conducted in the area of fraction multiplication (Izsák, 2008), but it has not yet been done in the area of fraction division. While investigating teachers' mathematical knowledge during a professional development course is valuable, the teachers might not use the knowledge that they used in the course to teach their students in their classroom because of many other issues around the classroom. In the professional development course, as part of the Does it Work data collection process, I had a chance to observe several teachers teaching their students fractions. One of the teachers, who performed well throughout the PD classes and interviews in terms of making sense of the various division situations, understanding other teachers' solution methods, and elaborating his thinking, did not utilize what he experienced in the course as much as other teachers whose performance in the PD classes and interviews was weaker. It might be due to his belief that his students would learn better or be enough with algorithms rather than with drawn representations. On the other hand, another teacher tried to utilize what she had learned through the course in her teaching, but her weak content knowledge prevented her from teaching effectively. Hence, it would be also worthy of exploring inter-connections among cognitive structures (e.g., knowledge, beliefs, or goals) related to their classroom, through investigating, to what extent, and why the teachers' and their students' interpretations were different or similar each other.

Author Note

This article is based in part on the author's doctoral dissertation (Lee, 2010), which was completed at the University of Georgia under the supervision of Denise Spangler. The author wishes to thank Leslie Steffe, Andrew Izsák, Chandra Orrill, and Jaehong Shin for comments that influenced the development of the ideas in the dissertation.

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