Are Teachers Amenable to Increasing Students’ Scope of Work in Doing Proofs? Estimating Teachers’ Decision-Making Using a Diagnostic Classification Model

Inah Ko*, Patricio Herbst**†

*Postdoctoral Research Fellow, University of Michigan, USA
**Professor, University of Michigan, USA

ABSTRACT

We investigate teachers’ decision making in contexts where they could choose to provide students more authentic experiences with proving. Specifically, we investigate their preferences to depart from norms about what proof problems to assign to students. Scenario-based instruments consisting of two sets of items reflecting two hypothesized norms in doing geometric proofs, the given and prove norm and the diagrammatic register norm, were used to operationalize teachers’ preference to depart from instructional norms in order to increase students’ share of labor. By applying a diagnostic classification model (DCM) to classify teachers with respect to their departure from two norms, this study shows that teachers’ decisions depend on the norm at issue. To examine individual factors associated with preference profiles, we use scores of teachers’ mathematical knowledge for teaching, beliefs on the importance of student autonomy, and confidence in mathematics teaching. This study also illustrates methodological benefits of a DCM model in estimating a binary construct (i.e., teachers’ preference), which has more than one sub-construct, with a small number of items.

Key Words: Doing proofs, Instructional situations, Scenario-based assessments, High school geometry, Teacher decision making, Instructional norm, Diagnostic classification model

INTRODUCTION

High school students in the United States have had consistent opportunities to learn to do proofs for more than a century, in the context of the high school geometry course (Herbst, 2002). These opportunities, however, have allocated a rather narrow scope of work to students. This narrow scope of the work can be explained by a set of norms that characterize what is expected from teacher and students in relation to the statement of geometric proof problems and the work doing those proofs (see Herbst, Chen, Weiss, & González, 2009). Norms are statements made by an observer that describe tacit regulations of social action. In this study, we focus on two particular norms called the given and prove norm (hereafter, DP-GP norm) and the diagrammatic register norm (hereafter, DP-DR norm). The given and prove norm states that it is usually the teacher or the textbook who decides what proposition students will prove. According to the studies of Herbst, Aaron, Dimmel, and Erickson (2013) and Herbst, Shultz, Ko, Boileau, and Erickson (2018), teachers recognize it as
normative for themselves to be the ones who state what is given and what is to be proven. On the other hand, the diagrammatic register norm describes the practice of stating proof problems in terms of particular diagrams. Specifically, it states that the problem for doing proofs is usually provided with a reasonably accurate diagram with labeled points and relies on the diagram to convey properties of incidence, betweenness, and separation. The norm also states that the diagram may not be relied upon for properties of parallelism, perpendicularity, and congruence but those properties must be stated explicitly (both what is given and what is to be proven) using the labels in the diagram to refer to the specific objects in the diagram (as opposed to referring to the concepts instantiated in the diagram; Herbst, Kosko, & Dimmel, 2013).

Herbst, Dimmel, and Erickson (2016) have shown that teachers recognize as normative to state proof problems using a diagrammatic register. The two different ways of stating the same problem, shown Figure 1a and 1b below illustrate examples of complying with and departing from this diagrammatic register norm (DP-DR). The case shown in Figure 1a shows compliance with the diagrammatic register norm while Figure 1b shows departure from the diagrammatic register norm in at least two ways: (1) the diagram has not been labeled and 2) the statement of the problem has been made in terms of concepts rather than diagrammatic objects.

While problems that abide by such norms for who states the parts of the problem and how those problems are stated offer some opportunities for students to do proofs, they are also limiting if we contrast the practice of proving they enable with the practices of mathematicians. As De Villiers (1990) has argued, one function of proof in mathematics is the discovery of new truths, but it is hard for students to use proof for that purpose if they are always told what proposition they will prove or which givens they must use to prove a desired conclusion. Likewise, translating statements formulated in terms of mathematical concepts (e.g., an isosceles trapezoid) to formulations about mathematical objects (viz., quadrilateral $ABCD$ with $BC \parallel AD$ and $AB \cong DC$), to the extent that it calls for managing the multisemiotic nature of mathematics (O’Halloran, 1998), is an important aspect of the mathematical practice into which students need to be inducted. In our work, we have been interested in gauging the extent to which teachers are amenable to departing from those norms and in understanding what accounts for that variability.

To examine the variability in teachers’ preference to depart from the norms, we have used scenario-based instruments (Herbst, Chazan, Kosko, Dimmel, & Erickson, 2016; see also Lieven, Peeters, & Scholaert, 2007) in which participants view depictions representing classroom situations and are presented with a decision for what to do next, in which they decide whether to follow a normative action or an action that departed from the norm. In our effort to understand the variability shown through teachers’ responses, we propose that the type of norm (e.g., DP-GP: given and prove norm, DP-DR: diagrammatic register norm) is a main source contributing to the variation in teachers’ decisions, not only among teachers but also within individual teachers. Specifically, we hypothesize that

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individual teachers’ decisions within a certain context may vary according to which norm is at issue.

The hypothesis that a contextual factor (i.e., which norm is at issue) systematically accounts for variation in the preference of an individual and among individuals is based on the broader hypothesis that there are tacit expectations that regulate actions of teachers and students in instances of an instructional situation; those expectations are described by the norms stated above. As different norms reflect different expectations for actions within an instructional situation, it is possible that teachers’ preference to depart from norms would depend on the norms at issue. For example, a teacher who chooses to do an action that complies with the norm of the diagrammatic register (e.g., providing an accurate diagram) may decide to choose a non-normative action (e.g., letting students decide a given of a problem) which departs from the given and prove norm. Moreover, as the norms are socially recognized in a group of experienced participants, we hypothesize that the influence due to the difference in norms on teachers’ preference would appear at a group level rather than solely at an individual level. In other words, although individual teachers differ in their actions depending on what they know or believe, there would be some similarities in what they prefer to do in regard to the two different norms. Thus, we use different norms to identify groups of teachers who have similar preference profiles with respect to the two norms.

While norms are characteristics of a social context (an instructional situation, e.g., doing proofs; see Herbst, 2006) and are meant to describe expectations of all participants of the context, the decision about whether to abide by a norm or not is still made at an individual level. In other words, even though there may be an overall pattern in teachers’ preference due to the norms (e.g., preference to comply with the DP-DR norm is distinguishable from preference to comply with the DP-GP norm), individual teachers might still differ as to whether they are disposed to depart from none of the norms, either of the norms, or both of the norms. We hypothesize that this individual variation regarding their preference profiles is associated with personal resources such as mathematical knowledge and beliefs. For example, teachers who possess more reform-oriented beliefs may prefer actions that depart from norms more than other teachers. Similarly, teachers who have stronger mathematical knowledge may prefer actions that depart from norms more than teachers who have weaker mathematical knowledge. Accordingly, in this study, we used teachers’ personal resources to examine the extent to which the difference among different groups of teachers is associated with the difference in their level of knowledge or beliefs on mathematics teaching and learning.

The investigations of our hypotheses on the characteristics of teachers’ preference to depart from the two norms and its association with teachers’ personal resources were guided by the following research questions:

1. Can we estimate teachers’ preference to depart from two different instructional norms, which are assumed to be binary (comply or depart from), using scenario-based survey items?
2. To what extent can some of the variability in the preference to depart from instructional norms be explained by which norm is at issue?
3. To what extent can the difference in the preferences among groups of teachers classified by the pattern of their preferences be explained by their mathematical knowledge for teaching geometry and beliefs on mathematics instruction?

THEORETICAL BACKGROUND

1. Factors influencing teachers’ decision-making

In an effort to understand the decisions that teachers make in classrooms, researchers have modeled teacher decision making with attention to individual or contextual resources (Schoenfeld, 2010; Brown & Coles, 2011). Researchers have seen teachers’ actions as projection of what they know, what they believe, and what they are oriented toward (Stahnhke, Schueler, & Roesken-Winter, 2016). Schoenfeld (2010) proposed a framework including teachers’ knowledge, beliefs, and goals as individual resources affecting teachers’ decisions. For example, Schoenfeld (2008) explained the decisions teachers make in teaching as a function of the teachers’ knowledge, goals, and beliefs by showing similarities in actions and decisions made by different teachers who have similar instructional goals but taught different classes regarding the students, content, and classroom styles. Among the researchers focusing on teachers’ knowledge, Dunekacke, Jenflén, and Blömeke (2015) modeled individual teachers’ mathematical content knowledge as a predictor of appropriateness of actions that the teachers plan for their students’ learning. Their result, however, showed that teachers’ planning of action cannot be directly predicted by teachers’ knowledge, but can be predicted through teachers’ ability to perceive mathematically important situations.
This result and other studies showing the effect of teachers’ situation-specific skills on teachers’ decision-making (Stahnke, et al., 2016) imply the need for consideration of context-specific factors to understand what makes teachers choose a certain action in their teaching.

Illustrating attention to contextual resources, Herbst and Chazan (2012) focus on the influence of instructional norms, which are described as “sociotechnical resources that a teacher makes use of when making decisions as to what to do” (p. 2). Our assumption about the association between instructional norms and teachers’ decision making in teaching (i.e., whether to take a non-normative action) is based on the theory of practical rationality. According to this theory, an instructional situation can be described by norms which specify mathematical elements of student tasks, actions called forth from the teachers and students, and the knowledge the teacher can attribute to students upon completion of the task. These norms apply to transactions about content, and Herbst and Chazan (2012) hypothesize that, although those norms may be tacit expectations rather than explicit rules, they are activated for specific transactions of content such as solving equations in algebra or doing proofs in geometry. If teachers’ decisions of what to do are made in an instance of an instructional situation, that is, regulated by certain norms, we conceptualize teachers’ preference with respect to a norm as an inherent characteristic of the construct, i.e., teachers’ preference to depart from the norms of the situation. In other words, we conceptualize teachers’ decision making (preference to do something) as a multidimensional construct consisting of norm-specific dimensions. In this study, we hypothesize two particular norms - given and prove norm (DP-GP) and diagrammatic register norm (DP-DR) - to measure teachers’ preference for non-normative actions in the instructional situation doing proofs. Following the hypothesis, we model teachers’ preference as a binary construct composed of two sub-constructs: 1) preference to breach DP-GP; 2) preference to breach DP-DR.

2. Departures from the two norms in doing proofs

We focus on the two particular norms in the instructional situation of “doing proofs.” Within the situation of doing proofs, students demonstrate their deductive reasoning skills and knowledge of theorems by engaging in proving specific claims. The first norm, *given and prove* or DP-GP, is that it is the teacher (or the textbook) who determines ‘given’ and ‘prove’ statements when assigning proof problems. The second norm, DP-DR, is that proof problems are presented using a diagrammatic register, that is, using geometric language connected to a specific geometry diagram. In the following, we provide examples of proof problems illustrating what departures from those norms could look like. We also describe the potential of the departures in expanding students’ scope of work in proof problems.

Cirillo and Herbst (2012) provided several examples of how a teacher might depart from the DP-GP norm or the expectation that they should be the ones spelling the givens and the conclusion to be proved. A teacher could, for example, ask students what would need to be assumed as true if they were to prove a particular conclusion (see Figure 2a). A teacher could also provide a set of givens and ask students what they would be able to prove using those givens (see Figure 2b). These two types of problems have the potential to engage students in proving as a means to discover what could be true rather than just to verify that something is true. Additionally, these problems could provide a context for students to experience the sense to which some assumptions are stronger than others, that some conclusions are stronger than others, and that some propositions are stronger than others (Mason, 1989). To engage students in these activities can help better connect activities of conjecturing and proving, which we have evidence to suggest are experienced as rather disconnected for students and teachers (Aaron & Herbst, 2019; Bieda, 2010).
Departures from the diagrammatic register norm could take many forms, depending on what aspect of the norm (or what subnorm) is targeted for departure (see Figures 3, 4, 5, 6, and 7). The first subnorm is related to Manders (1995) distinction between exact and co-exact properties of geometric diagrams: Manders (1995) called exact properties those that would be altered if the diagram was stretched (e.g., congruence, parallelism, perpendicularity, similarity) and co-exact those that would remain when the diagram was stretched (e.g., collinearity, incidence, separation, betweenness, continuity). Both in the Euclidean diagram and in the diagrammatic register, it is customary for co-exact properties not to be stated explicitly in symbols or words, but rather to be communicated by the diagram. Note that this is not the case in Hilbert’s geometry, where all properties are stated explicitly. Figure 3 shows an example of a proof problem breaching the first subnorm in that the statement of the problem explicitly describes properties of intersection ($AC, DB, XY$ be segments intersecting at $P$) and collinearity ($X$ and $Y$ are on $AD$ and $BC$), which may not be explicitly stated in a normative problem.

In Figure 4, we show an example of a proof problem where the second subnorm, that a diagram will accompany the problem statement, has been breached. The problem alludes to a generic object that has not been drawn. Clearly, students could draw the diagram, but it would involve some extra work translating from generic objects to diagrammatic objects. The third subnorm of the diagrammatic register states that the diagram includes labels for the points that need to be used in the proof sought. Figure 5 shows an example of a proof problem in which this subnorm has been breached, as point $O$ has been labeled but does not need to be used in order to prove that the triangle is isosceles. One could suspect that adding the label $O$ might suggest to students that triangles $AOM$ and $CON$ would need to be considered. We expect that while most teachers would label the other points, few would consider labeling $O$. Figure 6 shows an example of a proof problem in which the fourth subnorm of the diagrammatic register has been breached: The statement of the problem refers to the concepts in play (e.g., a right triangle, two of three given points, the midpoint of the segment formed by two of them) rather than to the diagrammatic objects that represent those concepts (viz., $\triangle ABC$, $A$ and $B$, $D$). Students who received a proof problem stated as in Figure 6 would need to engage critically in reading the statement. Furthermore, one could consider departing from the second, third, and fourth subnorm by only giving the proof problem statement as in Figure 6 but not including any diagram. Students would then have to draw the diagram themselves and decide which two points to use to determine the midpoint alluded to in the problem. Understandably, depending on how they chose where to locate the midpoint, this choice could lead them to draw a diagram where either the givens would not look like they are true (e.g., the points don’t look equidistant from the midpoint) or the conclusion does not look like it is true (e.g., the triangle does not look like a right triangle).
ARE TEACHERS AMENABLE TO INCREASING STUDENTS’ SCOPE OF WORK IN DOING PROOFS?

To have students deal with proof problems that depart from the diagrammatic register can also help toward getting them involved in a mathematical practice of wider scope. The diagrammatic register norm usually shelters students from having to engage in translating from concepts to objects and from objects to concepts. While the generality of mathematical propositions is better appreciated when they are stated in terms of concepts (e.g., base angles of an isosceles triangle are congruent), the proof and specific use of those properties often requires translation into objects (e.g., Figure 6). On the one hand, this translation can infuse precision in language (e.g., in Figure 5, one would talk about sides $\overline{AB}$ and $\overline{CB}$ being the congruent sides that make $\triangle ABC$ isosceles as opposed to saying two sides or those sides). On the other hand, this translation can allow for inference and in some cases formal calculation. After having named $\triangle ABC$ as the isosceles triangle and $\overline{AB}$ and $\overline{CB}$ as its congruent sides, it is logically warranted to define base angles as the angles of a triangle opposite to (i.e., not adjacent to) the congruent sides: If $\overline{AB}$ and $\overline{CB}$ are the congruent sides, $\angle BCA$ and $\angle BAC$ are their respective opposite angles, and thus they are the base angles. Thus, the expression “the base angles are congruent” can be recognized as a statement about two well defined angles. Without the translation of isosceles triangle to $\triangle ABC$ and the specification of which sides are congruent, it might be much less apparent whether base angles are well defined.

As such, departing from the norms could encourage students to have deeper understanding of, and value for, proving by expanding their engagement in doing proofs. Specifically, problems departing from the DP-GP norm can engage students in the activities connecting conjecturing and proving, which is usually not the case in normative problems (Aaron & Herbst, 2019). Similarly, problems departing from the DP-DR could serve to engage students in either translating or critiquing the way in which a conceptual argument establishes truth with precision.

However, the possibility that these activities engage students in valuable mathematical work does not entail that teachers will do so just by knowing what they could do. Our hypotheses on when teachers would be more likely to attempt to depart from the norms have motivated this study. A better understanding of the factors associated with teachers’ preference toward non-normative actions is important.
as it informs the kinds of resources or supports teachers may need to take actions that depart from the norm when the departure promotes student learning.

**METHODS**

1. **Participants**

The data analyzed in this study includes 472 U.S. high school (across 47 states) mathematics teachers’ responses on 1) seven scenario-based items measuring teachers’ dispositions towards the two norms, 2) four items measuring teachers’ mathematical knowledge for teaching doing proofs in geometry, and 3) 12 beliefs items designed by Stipek et al. (2001) addressing specifically the scales they called *confidence in teaching math* (five items) and *teacher control versus some child autonomy* (seven items). All the questionnaires including items asking teachers’ educational background and teaching experience were administered through the LessonSketch online platform (www.lessonsketch.org). *Table 1* includes descriptive statistics of the participating teachers’ educational background and their teaching experience.

As shown in *Table 1*, on average, participants had been teaching mathematics for 14.6 years (SD=8.6, min=1, max=40), and had taken 14 college-level mathematics courses (SD=7.5, min=2, max=40). In addition, participants had been teaching geometry for 5.9 years, on average.

<table>
<thead>
<tr>
<th>Table 1. Descriptive statistics for teachers’ background</th>
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<tbody>
<tr>
<td><strong>M</strong></td>
</tr>
<tr>
<td>Gender</td>
</tr>
<tr>
<td>Total years of teaching</td>
</tr>
<tr>
<td>Number of college math courses taken</td>
</tr>
<tr>
<td>Years of teaching geometry</td>
</tr>
</tbody>
</table>

2. **Decision instrument**

Each item in the decision instrument begins with a teaching scenario in the form of a storyboard that depicts a high school geometry class in which the teacher is posing a proof problem (see Herbst et al., 2016). Each of the items provides participants with four possible actions and asks to choose the one that best resembles what they would do next. Among the four given options, only one option represents an action complying with the targeted norm (DP-DR or DP-GP) and other three options represent an action departing from that norm (*Figure 8*). We coded...
teachers’ responses as binary: 0, if a teacher chose the normative action and 1, if a teacher chose one of the actions departing from the norm. A sample item measuring preference toward DP-GP is presented in Figure 8. According to our coding scheme, 0 is assigned to the response “a”, where the teacher provides ‘given’ and ‘prove’ statements and 1 is assigned to the responses “b”, “c”, and “d”, where the teacher allows the student to participate in setting up the proof problem. Among the total seven items, four items were designed to measure preference toward DP-DR and three items for DP-GP.

3. Knowledge instrument

Teachers’ mathematical knowledge was operationalized by a set of items measuring teachers’ subject matter knowledge for teaching high school geometry. To measure the aspect of the knowledge involved in the two norms in doing proof, we used items that measure teachers’ subject matter knowledge used in the task of choosing givens for a problem in the instructional situation of doing proofs. Choosing the givens for a problem is a recurring task of teaching that calls for teachers’ use of mathematical knowledge for teaching (Ko & Herbst, in press). The task choosing givens for a problem is related to the DP-GP and DP-DR norm in that the task includes creating or choosing the information such as diagrams, ‘given’, or ‘prove’ statements, to be included in the proof problems students are asked to solve. All the items were situated in classroom contexts and the specificity of the items regarding its intended aspect of the knowledge and their psychometric properties were studied by Ko (2019). Among the items, three multiple-choice items were dichotomously coded as “0” for incorrect responses and “1” for correct response. One testlet item (i.e., several binary questions dependent on a common stem) was coded by using the sum of the multiple true/false binary scores.

4. Beliefs instrument

Teachers’ beliefs and confidence related to mathematics teaching were operationalized by two subsets of the survey items developed by Stipek, Givvin, Salmon, & MacGyvers (2001): five items measuring “teachers’ beliefs about who should control students’ mathematical activity” and seven items measuring teachers self-confidence in mathematics teaching” (Stipek et al., 2001, p. 213). Each item was asked using a 6-point Likert scale ranging from “strongly disagree” to “strongly agree”. For example, an item of the first set asked teachers to indicate their level of agreement with the statement, “I believe that students should construct many of their own mathematics problems.” Similarly, an item of the second set asked participants to indicate their agreement on the statement, “I feel confident that I understand the mathematics material I teach.” A high score in the first set of items reflected strong belief in student autonomy (vs. teacher control) and a high score of the second set of items reflected high self-confidence in mathematics teaching.

Among several dimensions of teachers’ beliefs measured by Stipek et al. (2001), teachers’ beliefs on student autonomy was selected to be measured in this study, as the DP-GP and DP-DR are the norms that characterize the share of student work and departing from those norms would require expanding the scope of students’ work by giving them responsibility for making some choices (e.g., what to prove or how to represent the geometric objects involved). Similarly, teachers’ self-confidence with mathematics was selected among other dimensions as we hypothesized that whether teachers prefer non-normative actions or not could depend on how much confidence they have in themselves for managing instruction in which the students’ scope of work is expanded.

ANALYSIS AND RESULTS

1. Assumption on the scale of teachers’ preference

In a previous study of decision making, Herbst et al. (2016) presented the possibility to gauge teachers’ preference to comply with an instructional norm in separate decision items, by comparing each alternative to the normative choice, using multinomial logistic regression. In this approach, teachers’ responses on a single item were used as an outcome variable predicted by external variables such as norm recognition score or knowledge score derived from other instruments. A problem with that approach was that the analysis produced a number of paired comparisons about a specific decision, without a clear path toward aggregating those comparisons into scores that could inform teachers’ preference to comply with a norm across different items where the same norm could be activated. Therefore, the shared variance among the items designed to measure the same latent construct (i.e., teachers’ preference to depart from a specific norm)
influencing teachers’ responses across multiple items could not be examined. Accordingly, Erickson, Dimmel, Herbst, & Ko (2015) scaled teachers’ normative dispositions using correlations among the multiple items designed to assess decisions for a given norm. However, the use of a small number of items with the assumption that the scale of teachers’ preference is continuous resulted in low reliability of the individual scores. Also, classifying groups of teachers with similar preference patterns across the two norms was not sufficiently warranted with continuous scores as it requires subjective decision on the thresholds between groups.

Our present study attempts to overcome those limitations by reconsidering the assumptions the previous studies had made about the construct – teachers’ preference in decision making – being continuous. Given the characteristics of teachers’ responses and theoretical assumptions on the construct, for example, decision makers tend to associate their conceptions of the acts with a particular choice (Tversky & Kahneman, 1981), we suggest an alternative assumption. The assumption is that teachers’ preference for a decision is a binary construct – that is, that the construct indicates whether teachers favor or don’t favor normative actions. Based on this assumption, we use a diagnostic classification model (Rupp, Templin, & Henson, 2010; Bradshaw, Izsák, Templin, & Jacobson, 2014), which enables modeling binary constructs and classifying teachers based on their preference patterns across DP-DR and DP-GP norms.

This model assumes that a set of categorical latent constructs influences participants’ responses to items. The categorical latent constructs are called attributes and each attribute is generally measured with two categories: 1) a participant possesses the attribute or 2) a participant does not possess the attribute (Jurich & Bradshaw, 2014).

2. Preference profiles

To estimate teachers’ preference to depart from the two norms, we chose a log-linear cognitive diagnosis model (LCDM) from among several types of DCMs. Given that a LCDM allows item parameterization to vary at the item level (Henson, Templin, & Willse, 2009), we chose this model for the possibility that different instances of a situation offer different alternatives to the normative one, and yet items may all contribute to determine the likelihood that a teacher would prefer to depart from the norm. This choice was especially appropriate considering that the decision items for each norm varied in regard to how the preference to depart from the norm was being tested (e.g., in some DP-GP items the teacher asked the students to suggest the givens while in others the teacher asked the students what could be proved). Following the correspondence between items and targeted attributes, the LCDM model was set to estimate the main effects of items on each of the two norms–DP-DR and DP-GP. The Q-matrix specifying the item-to-norm alignment is presented in Table 2.

<table>
<thead>
<tr>
<th>Item</th>
<th>Attribute 1: DP-DR</th>
<th>Attribute 2: DP-GP</th>
</tr>
</thead>
<tbody>
<tr>
<td>D61011, D61012, D61013, D61014</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>D62011, D62013, D62014</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The formula below expresses a generalized linear model with a logit link (the LCDM) established for estimating teachers’ preferences to breach the DP-DR norm (or DP-GP norm, depending on the norm the item is designed to measure).

\[
\ln\left( \frac{p(X_i = 1 | \alpha_{ra})}{p(X_i = 0 | \alpha_{ra})} \right) = \lambda_{i,0} + \lambda_{i,1}(a)\alpha_{ra},
\]

where \( a = \text{DP-DR or DP-GP} \)

In the equation, \( X_i \) indicates a dichotomous item response to item \( i \) (for \( i = 1, \ldots, 7 \)) by a person whose latent class is \( \alpha_{r} \). \( \alpha_{r} = [\alpha_{r,DP-DR}, \alpha_{r,DP-GP}] \) is a vector of attribute profiles (\( \alpha_{r,DP-DR}, \alpha_{r,DP-GP} \in \{0,1\} \)) and the dependent variable of \( \ln(\frac{p(X_i = 1 | \alpha_{ra})}{p(X_i = 0 | \alpha_{ra})}) \) is the logit of a non-normative choice in item \( i \) by respondents who have an attribute profile \( \alpha_{r} \). Specifically, the LCDM model estimates the item parameters \( \lambda_{i,0} \), indicating the log-odds a participant whose preference is to apply the norm chooses the non-normative option, and the main effect \( \lambda_{i,1}(a) \), indicating the increase in log-odds of a non-normative choice for teachers whose preference is to depart from the norm. Using these parameters, we calculated the probabilities of choosing a non-normative option for each item (Figure 9). Given the large difference in the probability of choosing a non-normative option for each item (Figure 9),
normative action between participants who prefer to comply with the norm and against the norm (ranges from 23% to 43%), all the items acceptably discriminate between participants with different preferences to the norms. All the analysis was done using Mplus (Muthen & Muthen, 1998-2015). The model fit was evaluated using a bivariate goodness of fit statistics with a $\chi^2(1)$ distribution for each pair of items (Rupp et al., 2010). The result suggested that the estimated LCDM model across two norms provided an acceptable model fit (no item pair showing misfit at a 0.05 significance level).

According to the estimated attribute profile, participants could be classified into one of four profile groups: 1) preferring to comply with both DP-DR and DP-GP, 2) preferring to comply with DP-DR, but preferring to depart from DP-GP, 3) preferring to depart from DP-DR, but preferring to comply with DP-GP, and 4) preferring to depart from both DP-DR and DP-GP. The proportion of teachers who are classified with each of the preference profiles based on the estimated model is presented in Table 3.

As shown in Table 3, 18% of teachers preferred to comply with both norms and 60% of teachers preferred to depart from both norms. This high proportion of teachers (78%) who have the same preference to both norms indicates a high correlation between the two norms. This high correlation between the two norms is understandable in that all of the items were situated in the same instructional situation of doing proofs. The association is, however, not perfect given that the model captured noticeable proportions of teachers who are estimated as having a non-normative preference only for one of the norms. In particular, there were 18% teachers who preferred to comply with the DP-DR norm but not with the DP-GP norm, whereas there were only 4% teachers who preferred to comply with the DP-GP norm but not with DP-DR norm. This implies that a higher proportion of the population is estimated to prefer to depart from the DP-GP norm than the DP-DR norm. This proportion is called a base rate, which can be calculated by taking the sum of the probability distribution for the profiles that include the presence of departing each norm in teachers’ preference.

Table 3. Proportions for the preference profiles based on the estimated model

<table>
<thead>
<tr>
<th>Profile</th>
<th>Estimated proportions of teachers</th>
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</thead>
<tbody>
<tr>
<td>Comply with both norms</td>
<td>18%</td>
</tr>
<tr>
<td>Comply with DP-DR &amp; Depart from DP-GP</td>
<td>18%</td>
</tr>
<tr>
<td>Depart from DP-DR &amp; Comply with DP-GP</td>
<td>4%</td>
</tr>
<tr>
<td>Depart from both norms</td>
<td>60%</td>
</tr>
</tbody>
</table>

This higher base rate of preference to depart from DP-GP (78%) than DP-DR (64%) (Figure 10) indicates that DP-GP is a norm easier to depart from.
than DP-DR. In other words, it is less likely for teachers to take an action that departs from the expectation that the proof problems be presented in a diagrammatic register than to take an action that departs from the expectation that the teacher is who determines the given and proof statements. This difference could be interpreted in terms of the complexity of managing the situation of doing proofs into departing from the DP-DR norm and the DP-GP norm. If a teacher departed from DP-DR, students could do very divergent activities depending on how they draw and label the diagram. Departing from DP-GP could be relatively less complicated for teachers to manage, as eventually a teacher might consider what each student did as a different problem (and not as the same problem with different representations; see also Herbst et al., 2018).

Each score of teachers’ knowledge and beliefs was generated from a unidimensional model in which latent factor variances and means were set to 1 and 0, respectively. Higher scores indicated higher level of mathematical knowledge, stronger beliefs on the value of student autonomy, and higher confidence in mathematics teaching. All the unidimensional models yielded good model fit statistics (RMSEA<0.05, CFI>0.95, TLI>0.95), indicating that each set of items coherently and significantly contributes to the measure of the knowledge and beliefs (autonomy and confidence, respectively). The averages of the scores were then compared across the four groups of teachers classified by the pattern of their preference to depart from DP-DR and DP-GP norm. Figure 11 presents the differences in group means of MKT, student autonomy, and confidence.

As shown in Figure 11, the group of teachers who prefer to depart from both norms had higher scores of MKT, beliefs about student autonomy, and confidence in mathematics teaching than other groups who prefer to comply with either or both of the norms. This result aligns with our hypothesis that teachers who have stronger mathematical knowledge and confidence in teaching would be more likely to depart from the norms than others who have less knowledge and confidence as they might be more confident and better prepared for managing instruction that deviates from expected situational norms. However, despite this noticeable characteristic of the profile group preferring to depart from both norms, the differences in MKT and confidence are very small and not statistically significant, so we interpret that how much MKT or confidence teachers had was not a significant factor associated with their preferences to depart from either or both of the norms in this study. However, with unbalanced sample sizes, caution must be applied, as the small sample size (e.g., nine teachers in the group “Depart from DP-DR & Comply with DP-GP”) may not be enough to represent the characteristics of the group.

In contrast to teachers’ MKT and confidence, the four groups were significantly different from each other in their beliefs about student autonomy in classroom lessons. Specifically, the group of teachers who prefer to depart from both of the norms showed significantly higher degree of agreement on the statements emphasizing the value of allowing student autonomy in classrooms than other groups who prefer to comply with either or both of the norms. For example, the differences between the group “depart from both norms” and the

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Figure 10. The estimated proportion of teachers’ in the population who prefer to depart from the norm (base rate)

3. Relationship between profiles and teachers’ individual characteristics

To examine the extent to which the difference in the preferences among groups of teachers classified by the pattern of their preferences is associated with their knowledge and beliefs, we compared the averages of three factor scores reflecting teachers’ mathematical knowledge for teaching (MKT), beliefs on students’ autonomy, and confidence in mathematics teaching, respectively, across the four profile groups. The four profile groups were identified by using the variable representing teachers’ most likely latent class membership estimated by our DCM model.

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2 Teachers’ MKT was measured by four items designed to measure teachers’ subject matter knowledge used in the task of ‘choosing givens for a problem’ in the instructional situation of doing proofs in high school geometry.
groups “comply with DR & depart from GP”, “comply with both norms”, “depart from DR & comply with GP” were 0.30 (p<0.01), 0.58 (p<0.001), and 1.16 (p<0.001), respectively. This result is aligned well with our expected association between teachers’ preference and their beliefs on student autonomy in that the actions that we present in our items deviate from normative actions regarding the scope of students’ work in classrooms. The presented teachers’ actions were non-normative in that they expanded students’ role in the work of proving to more than what is expected in normative actions and enabled students to make more choices than they normatively make. Thus, the result is understandable given that teachers who value giving students a greater share of labor and less teacher control would be more likely to prefer to depart from the normative actions that limit the scope of student work in doing proofs.

In summary, the results examining the association between the profiles and teachers’ individual resources suggest that there is a significant association between teachers’ profiles and their beliefs about the importance of encouraging students to have some autonomy in classroom lessons. However, clear evidence of the association between teachers’ MKT or confidence and their profiles could not be identified in this analysis.

DISCUSSION AND CONCLUSION

This study illustrates how an application of diagnostic classification modeling can help gauge differences between teachers’ preferences toward departing from different instructional norms and how their preferences are associated with their personal beliefs on mathematics teaching and MKT. The application of DCM as well as the results from the model have methodological and theoretical implications to the field of mathematics education.

Methodologically, the application of the DCM model shows that teachers’ preference to depart from the two different norms, DP-DR and DP-GP, can be reliably estimated under the assumption that each construct is binary. Specifically, a diagnostic classification model (DCM) could yield acceptable reliability for the estimates of teachers’ preference to depart from each norm, which had not been possible in a previous study that had examined items making

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3 The exact number of teachers for each group in our sample is slightly different from the proportion of teachers presented in Table 3, because the former is based on the most likely latent class membership assuming the perfect classification and the latter is the estimated proportion of teachers’ in the population based on posterior probabilities.
the assumption that the constructs were continuous. Also, the DCM model allowed criterion-referenced interpretations of teachers’ decision making across the two different norms: Teachers are more likely to follow the DP-DR norm than the DP-GP norm. The difference in teachers’ preferences for each of the norms could be interpreted with regard to the complexity in managing instruction that departs from the norm of an instructional situation. As illustrated in this study, researchers could consider a DCM model for a construct which can be presented by a binary scale when the purpose is to identify groups of participants with similar latent profiles across hypothesized multiple dimensions of the construct.

Theoretically, this study suggests that teachers’ decision making in an instructional situation can be multidimensional, depending on differential preferences to depart from different norms that regulate the same instructional situation. Hence, whether to take a non-normative decision or not can be dependent on the norm that is at issue. Specifically, a significant number of teachers (22%) who prefer to depart from only either DP-DR or DP-GP norm would not have been identified if teachers’ decision-making was operationalized as a single construct (e.g., preference to enact a normative instance of doing proofs) without a consideration of the contextual factor - types of instructional norms. It should be noted, however, that we are not arguing that teachers’ preference to depart from different norms is the only factor that influences specific teachers’ decisions. A teacher’s perception of unique aspects of the particular problem or their awareness of their students’ capabilities could matter in what an individual decides to do. However, in our study, the only trait that is common within a factor and is different between factors is the type of norm. Thus, it is reasonable to conclude that the systematic variance in teachers’ preference is explained by the type of norm. One unanticipated result was that no significant differences were found in teachers’ MKT and confidence among different profile groups. A possible explanation for these results is that the association between teachers’ decision and their MKT or confidence depends on what aspect of the norm is targeted for departure. As described in the introduction, each DP-DR and DP-GP norm consists of different subnorms. For example, Boileau, Dimmel, and Herbst (2016) studied relationships between teachers recognition of different aspects of the DP-DR norm (as illustrated above in Figures 3-7) and MKT-G scores measured by the set of items initially studied by Herbst and Kosko (2014); Boileau et al. (2016) showed that those aspects were distinguishable subdimensions of DP-DR norm and that teachers’ MKT-G scores were significantly associated with teachers’ endorsement of the 3rd DP-DR subnorm (i.e., the teacher assigns a proof problem with an accompanying diagram where all the points needed in the proof are labeled), but not with the 1st or 5th DP-DR subnorm (i.e., 1st: properties stated in the givens do not include co-exact properties, which are relied upon in the diagram, 5th: the diagram accurately represents the figure addressed in the problem). Considering that our DP-DR decision items target different sub-norms, the association between teachers’ MKT and an overall preference to DP-DR norm might have been confounded by the items measuring the subnorms not associated with MKT. To develop a full picture of the associations between teachers’ decisions and their MKT or confidence, additional studies measuring teachers’ preference specific to each subnorm may need to be conducted in the future. Another possible explanation for the nonsignificant impact of the MKT measure is that the MKT items used to test knowledge needed to create the givens for a proof problem may not have included enough items that sampled the use of knowledge in contexts that breach norms: Most items asked the respondent to choose or evaluate appropriate givens needed for proof problems that complied with the norms. Thus, to develop a full picture of the associations between teachers’ decisions and their MKT, it would be desirable to expand the set of items measuring MKT needed for managing an instructional situation to also include items that measure knowledge needed in instances where the proof problems presented in the MKT items deviate from the norms.

Finally, our results do show a significant relationship between beliefs on the value of students’ autonomy (as measured with the instrument developed by Stipek et al., 2001) and the preference for decisions to increase the scope of work for students by departing from both norms. It appears that teachers who value students’ autonomy are
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It appears that teachers who value students’ autonomy are willing to present proof problems in which students take charge of more aspects of the translation between the conceptual and diagrammatic registers.

Author Note

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