Journal of Educational Research in Mathematics
2020, Special Issue, 185 ~ 198

Korean Teachers' Mathematical Knowledge for Teaching in Algebraic Reasoning

Yeon Kim*

Professor, Silla University, South Korea

ABSTRACT

To collect information about teachers concerning mathematical knowledge for teaching and find out what needs to be considered in developing a curriculum to teach it, the current study surveyed 137 secondary teachers and interviewed thirteen of them in Korea. The survey and interviews used the assessment of mathematical knowledge for teaching about algebra I, which was developed as part of the Measures of Effective Teaching project. The correct response rate of Korean teachers was very high, but there were some differences found in the areas of algebraic reasoning. Furthermore, mathematical analysis is important in assessing students' algebraic reasoning, and each teacher's typical teaching method is formidable in evaluating students' reasoning. Implication is discussed for the improvement of teachers' mathematical knowledge for teaching algebra.

Key Words: Mathematical knowledge for teaching, Mathematical reasoning, The Measures of Effective Teaching project, Algebra, South Korea

INTRODUCTION

It has been reported that Korean mathematics teachers' level of mathematics knowledge is higher than that of some other countries' teachers (Blömeke & Paine, 2008; Schmidt et al., 2008). This high level of mathematics knowledge might guarantee that teachers do not offer mathematically incorrect facts to students and have enough knowledge to find correct solutions. It does not, however, address teachers' pedagogical approaches to teaching mathematics in secondary school. Moreover, little research has investigated how Korean secondary teachers teach mathematics and what mathematical knowledge for teaching they have. *Mathematical knowledge for teaching* (MKT) is "the mathematical knowledge used to execute the work of teaching mathematics" (Hill, Rowan, & Ball, 2005, p.373), and this knowledge is crucial to the improvement of teaching mathematics (National Mathematics Advisory Panel, 2008). Like other East Asian countries, the great importance attached to the college entrance examination leads to research and popular interests in secondary mathematics education focusing on students' higher scores more than a focus on the work of teaching performed in instruction or the quality of instruction.

Teaching mathematics is an intellectual activity which requires a variety of interactions between teacher and students in the mathematics classroom and entails professional knowledge and reasoning to invite students into the process of knowing mathematics (Ball, Thames, & Phelps, 2008). The perspective of teaching as consisting of interactions among human beings emphasizes helping students act as critical thinkers, developing their reasoning, and collectively working with their classmates in the mathematics classroom (Cohen, 2011). Such interactions are initiated and developed by the "work of teaching," such as interpreting students' reasoning, judging and correcting textbook treatments of particular topics, using representations accurately in the classroom, and providing students with examples of mathematical concepts, algorithms, or proofs (Hill et al., 2005). However, research on MKT is insufficient in

Email: yeonkim10@silla.ac.kr

Please cite this article as: Kim, Y. Korean teachers' mathematical knowledge for teaching in algebraic reasoning. *Journal of Educational Research in Mathematics* (2020). https://doi.org/10.29275/jerm.2020.08.sp.1.185

South Korea, in terms of exploring such topics as what teachers think about MKT, what MKT teachers have or do not have, and how MKT is learned with reliable surveys. This absence of research on MKT in South Korea has not created a research climate to focus on professional development regarding MKT.

The current study aims to examine features of Korean teachers' MKT, particularly about students' algebraic reasoning and to get some sense about how to approach professional development to offer better instruction to improve students' algebraic reasoning. By using the assessment of MKT for algebra as part of the Measures of Effective Teaching (MET) project (Phelps, Weren, Croft & Gitomer, 2014), the current study surveyed teachers about their MKT and conducted interviews in order to identify what decisions and actions teachers make in situations related to students' algebraic reasoning. The following question frames the research: What features do Korean teachers have about students' reasoning in algebra with regards to MKT? This question was investigated by two ways. First, based on 137 Korean teachers' responses to MET assessment, rates of correct responses were calculated and then analyzed by F-tests and the Pearson correlation. Second, based on interviews with thirteen Korean teachers' responses to the MET assessment, the features of teachers' responses were reported.

BACKGROUND

1. Mathematical Knowledge for Teaching

MKT is the mathematical knowledge needed to perform the work of teaching (Ball et al., 2008). MKT has been identified as significant in teaching mathematics (Lewis & Blunk, 2012; Hill, Umland, Litke, & Kapitula, 2012) and in student achievement (Rockoff, Jacob, Kane, & Staiger, 2011). Based on Shulman (1986), Ball et al. (2008) specified three subdomains within pedagogical content knowledge (knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum) and three subdomains within subject matter knowledge (horizon content knowledge, common content knowledge, and specialized content knowledge). Although these separate subdomains can be used for analysis, they are inevitably intertwined, continuous and spontaneous in the instructional practices (Koellner et al., 2007; Kim, 2016). Routines that teachers perform in teaching mathematics are based on all the features of their knowledge and reasoning that they have accumulated or learned throughout classroom instruction. Such performance includes presenting mathematical ideas, finding an example to make a specific mathematical point, appraising and adapting the mathematical content of textbooks, modifying tasks to be either easier or harder, giving or evaluating mathematical explanations, and asking productive mathematical questions (Ball et al., 2008).

To assess teachers' MKT, the Learning Mathematics for Teaching project (LMT) developed assessments in order to link teacher performance on the LMT measures to K-8 student performance on the standardized achievement test. They found that teachers' scores were significant predictors of students' scores (Hill et al., 2005). The LMT assessments have been studied in Ghana, Indonesia, Norway and Ireland as well, through validation studies that explore the utility of MKT measures developed in the U.S. (Cole, 2012; Delaney et al., 2008; Mosvold & Fauskanger, 2009; Ng, 2009). Delaney and his colleagues (2008) adapted the MKT measures for use in Ireland. They found that some Irish teachers were unsure of the meaning of certain terms and suggested changes related to the general cultural context for adaptation. Mosvold and Fauskanger (2009) found that distinct changes were needed in the translation from American English into Norwegian because some concepts in the former are not present in the Norwegian curriculum. They claimed that such changes could make the assessments more accessible to Norwegian teachers. Ng (2012) also found some contextual issues in Indonesia and differences in instructional practices and representations. Cole (2012) indicated issues such as the cultural incongruence of the item contexts between U.S. and Ghana.

In a similar work at the secondary level to develop assessments and assess secondary teachers' knowledge, McCrory and her colleagues (2012) developed a framework about the knowledge for teaching algebra. This framework includes the three domains of knowledge, that is, school algebra (knowing what teachers will teach), advanced mathematical knowledge (knowing more advanced mathematics that is relevant to what they will teach), and teaching knowledge (knowing mathematics that is particularly relevant for teaching). The framework also involves the other three categories of practices,: trimming (making mathematics accessible to students while retaining the integrity of the mathematical ideas); bridging (providing students with the big picture of mathematics); and decompressing (investigating the concepts of mathematics for teaching). The framework consists of the combination of the categories of knowledge and practices of teaching, and, thus, assessment items can be specifically written for each combination. McCrory et al (2012) emphasized a framework that lays out the major areas of mathematical knowledge for teaching algebra in order to develop measurements as well as investigate the relationship among teachers' knowledge, teachers' practice and students' learning. They claimed that such empirical support would be foundational for secondary school mathematics teacher preparation.

The MET project aimed to build and test measures of effective teaching to determine how evaluation methods could best be used to tell teachers more about the skills that make them most effective and to help districts identify and develop great teaching. As part of the MET project, three sets of assessments (Grades 4-5, Grades 6-8, and algebra) of MKT were developed in order to measure the types of mathematics knowledge used in teaching practice and then investigate a teacher's impact on student achievement (Phelps et al., 2014). The assessment design framework for this project concentrates on the content practices that teachers encounter in the range of settings and roles that define their work. The MET assessment measures the knowledge and reasoning embedded in teaching by using contexts which are legitimate and pertinent because "the tasks of teaching attempt to capture how teachers actually work with content in their day-to-day and moment-to-moment practice (Phelps et al., 2014, p. 8)." The current study uses the assessment for algebra I as a way to gather information about secondary teachers concerning MKT.

2. Teachers' Knowledge of Algebraic Reasoning

Algebra is a traditional content area in school mathematics that continues to be important today. It underlies teaching from kindergarten through 12th grades in Principles and Standards for School Mathematics (National Council of Teachers of Mathematics, 2000). Algebra is considered a gatekeeper subject for students' subsequent education and employment (e.g., Cai & Knuth, 2011; Izsák, Çağlayan, & Olive, 2009; Moses & Cobb, 2001). For such an important subject, teachers' conceptions and misconceptions about specific areas within algebra, mainly functions (e.g., Even, 1993; Hitt, 1994; Norman, 1992; Chinnappan & Thomas, 2001), slope (e.g., Stump, 2001), and covariance (e.g., Sherin, 2002), have been investigated.

Research has reported what algebra secondary teachers know and do not know. Tirosh, Even and Robinson (1998) investigated inservice teachers' understanding of students' conceptions and misconceptions about expressions, such as *3x+4* to get either *7x* or *7*. The two preservice teachers were unaware of this tendency, but the two experienced teachers were aware of it, and planned their lessons accordingly. Asquith, Stephens, Knuth & Alibali (2007) found that inservice teachers were good at predicting students' understanding of a variable as being an unknown number or having multiple values, although these teachers rarely predicted students' understanding of the equal sign, and rarely identified students' misconceptions about variable and the equal sign. Based on the surveys of and interviews with algebra teachers about the advantages and disadvantages of instruction that includes a focus on multiple strategies, Lynch and Star (2014) found that some secondary teachers do not recognize multiple strategies in instruction as a critical window to capture what misconceptions students might hold or as a useful or appropriate method to enable students to progress from intuitive to formal strategies. Postelnicu (2011) found that inservice teachers did not identify students' difficulties with slope and the Cartesian connection.

In further studies of preservice teachers, Huang and Kulm (2012) reported preservice teachers' lack of knowledge about the concept of function regarding school mathematics, teaching mathematics, and advanced mathematics. Specifically, the study illustrated teachers' weaknesses in representational flexibility and selection of function perspectives, and the numerous mistakes made in algebraic manipulation and function transformations. The authors recommended that the teacher preparation curriculum provide coherent content areas. Tanisli and Kose (2013) also found that, in general, preservice teachers had inadequate knowledge concerning what difficulties and misconceptions students have about the concepts of variables, equality, and equations. Caglayan (2013) investigated preservice teachers' understanding of polynomial multiplication and factorization and also recommended that the teacher education program provide opportunities for preservice teachers to develop their quantitative reasoning.

Teachers' limited understanding of the concepts in algebra has several negative ramifications. It reduces the kinds of tasks that teachers select for students to participate in, narrows and restricts the types of questions asked, and hinders the process of making connections within the curriculum (Haimes, 1996; Heid, Blume, Zbiek, & Edwards, 1999; Stein, Baxter, & Leinhardt, 1990; Wilson, 1994). Teachers may also emphasize the procedural aspects of the equation solving processes over the conceptual underpinnings (Attorps, 2003). However, teachers' well-connected under-standing about algebra is necessary to take a conceptual approach to the teaching and use representations in order to help students have meaningful discussions (Lloyd & Wilson, 1998; Chazan, 1999, 2000). Teaching a function-based, technology-intensive algebra curriculum appears to help a teacher develop new perspectives on equations and equation solving, but also reveals weak areas in a teacher's conceptual under-standing (Chazan, Larriva & Sandow, 1999; Chazan, Yerushalmy, & Leikin, 2008). Therefore, teachers need to have opportunities to learn algebra in and for teaching. Such learning opportunities for teachers are important because teachers generally do not have a relational understanding of various representations in algebra for teaching (Stump, 1999) and teachers cannot fill the gap between undergraduate courses in algebra and school algebra (Belfort, Guimaraes, & Barbastefano, 2001).

DATA AND METHODS

The current study used the assessment of MKT for algebra I as part of the MET project.[1] This assessment focused on the mathematics knowledge used in recognizing, understanding, and responding to the mathematical problems that teachers encounter as they teach mathematics. The assessment scores included both selected-response and constructed-response questions. To evaluate the appropriateness of the content of the assessment regarding the curriculum of South Korea, the original English version assessment, written by the MET project, and Korean-translated assessment were provided two external advisory researchers who have Ph.Ds. in mathematics education and are fluent in English and Korean. The reviews suggested excluding items 5, 6a and 22 in order to have assessment validity, given that items 5 and 22 are applicable only for English Language Learners and the content of item 6a is the FOIL method, which is unfamiliar to Korean teachers.[2] Option d of item 17 also includes FOIL, but it is used in an operation manner and an understanding of it is not as crucial as it is in item 6a in giving a correct response. Thus, the multiplication formula was substituted for FOIL in this item for the Korean version. The Korean translation was also reviewed throughout this external reviews to ensure that the translation did not result in any changes in the meanings of the original items. Interviews with the teachers indicated that most interviewees could not understand students' ideas in item 20, and, therefore that item was excluded in this study for the validity of the assessment. Because of the negative item-rest correlation and low Cronbach's alpha, item 3a, 3b, 6b, 6c and 19 were further excluded for the analysis. Ultimately, twenty-eight items among the original thirty-seven were analyzed in this study, consisting of eleven items about expressions, thirteen about equations, and four about functions. The final reliability of these items was 0.68, which is reliable according to the American Educational Research Association, American Psychological Association and National Council on Measurement in Education (1999).

This study surveyed 137 Korean teachers in secondary mathematics schools all over the country, who were recruited via voluntary participation. Because secondary teachers in South Korea can teach middle and high school students, there was no limitation in the eligibility of participation. Among the participants, 48% were teachers who were teaching middle school students at that time, and 37% were male. The teachers' teaching experience varied from one to 41 years, and its average was almost eight years. All of them had at least an undergraduate college degree and had a teacher certification for secondary mathematics. All teachers were in full-time positions, which is typical in Korea. The assessment was administered by the author as an untimed test, which ranged approximately between 30 and 60 minutes.

In parallel with the collection of the teachers' responses, the participants were invited to take part in

[1] Algebra I generally includes linear equations, linear inequalities, linear functions, exponential equations and functions, polynomial equations and factoring, quadratic functions, quadratic Equations, real numbers, and rational equations and functions, etc.

[2] FOIL is a mnemonic for the standard method of multiplying two binomials. The letters FOIL stand for First, Outer, Inner, Last. First means multiply the terms which occur first in each binomial. Then Outer means multiply the outermost terms in the product. Inner means multiply the innermost two terms. Last means multiply the terms which occur last in each binomial. Then simplify the products and combine any like terms which may occur.

individual interviews to probe their thinking about the assessment items, and thirteen of them volunteered. Among these thirteen participants, seven interviewees taught middle school students, and six interviewees were female. Their teaching experience varied from one to twenty-seven years, and its average was almost eleven years. The interviewees' scores of the assessment had descriptive statistics that were similar to all participants in terms of mean, standard deviation and range. The interviews helped explore how participants interpreted the items and their process of reasoning about them. In general, the interviewees first solved each item and then were asked retrospective questions to get information about the solution process (e.g., What did you get from the item? How did you figure this out? Why did you do that? Can you tell me what you were thinking?). All interviews were audio-recorded and transcribed.

This study reported rates of correct responses for each item of the assessment because it showed detailed information about the participants' response for each item. For example, 0.5 as a rate of correct response indicates that one half of the participants responded correctly to an item. The rates of correct responses are ranked in order to find items that Korean teachers found difficult. Because there are three content areas in the assessment—expressions, equations, and functions—*F*-tests were applied for each content area, respectively. The Pearson correlation coefficients were also measured between teachers' teaching experience and the total rates of correct responses and between teachers' teaching experience and each content area. The interview data was inductively analyzed by the constant comparative method (Strauss & Corbin, 2008). Specifically, while reviewing interviewers' responses, I considered what sources and reasoning the interviewers drew upon in answering each item. I sorted the interviewers' responses and sources and characterized them.

RESULTS

1. Great performance is shown, but it is not all

Rates of correct answers among the participants for each item are reported in *Table 1*. The average rate of correct responses was 0.80 (*SD*=0.21), which is high.1 However, three items that the participants performed poorly on, listed in order from the item with the lowest performance, are 12, 17, and 8. Their rates of correct responses are respectively 0.138, 0.379, and 0.445.

Table 1. Rates of correct answers for MET algebra I assessment

item number	The mathematical topic	Rates of correct responses
1	equations	0.788
2	expressions	0.948
3c	equations	0.919
3d	equations	0.824
3e	equations	0.927
4	equations	0.985
7	expressions	0.781
8	functions	0.445
9	equations	0.883
10	functions	0.729
11a	equations	1
11b	equations	0.956
11c	equations	0.948
11d	equations	0.963
11e	equations	0.963
12	functions	0.138
13	expressions	0.605
14	expressions	0.810
15a	expressions	0.941
15b	expressions	0.927
15c	expressions	0.941
15d	expressions	0.846
15e	expressions	0.737
16	equations	0.656
17	functions	0.379
18	expressions	0.978
21	equations	0.569
23	expressions	0.912

These items are about function and quadratic function and its representations—evaluating students' multiple reasoning about slope, evaluating students' reasoning about a graph, and evaluating problems about the shift of quadratic function, respectively.

A one-way between subjects ANOVA was conducted to compare the rates of correct responses for items about equations, expressions and functions. There was a significant difference in them at the $p<.05$ level for the three areas [$F(2, 408)=297.5$, $p < 0.001$]. Post hoc comparisons using the Tukey HSD test indicated that the mean score for functions (*M*=0.43, *SD*=0.24) was significantly different than equations (*M*=0.88, *SD*=0.11) and expressions (*M*=0.86, *SD*=0.13), respectively. However, the rates of correct responses for equations did not significantly differ from the rates of correct responses for expressions (p=0.21). Taken together, the Korean teachers who responded to the MET assessments were good at equations and expressions, but not at functions. Furthermore, the results of the Pearson correlation indicated that there was no correlation between teaching experience and overall rates of correct responses [$r(136)= -0.375$,

$p<0.001$]. Respectively, there was no correlation between teaching experience and rates of correct responses for equations [$r(136) = -0.262$, $p<0.05$], between teaching experience and rates of correct responses for expressions [$r(136)= -0.289$, $p< 0.001$], and between teaching experience and rates of correct responses for functions [$r(136)= -0.301$, $p<0.001$]. In total, teaching experience was not statistically related to the level of MKT.

2. Mathematical analysis is critical, but how to teach is formidable

Three features are prevalent in the interviews with thirteen teachers. First, mathematical analysis and decision is important in assessing students' reasoning whether or not teachers have experience with the student's work. As shown in *Figure 1*, item 16 shows a student's non-standard response about solving a quadratic equation which has two integer roots. After determining that the student's response does not have any mathematical errors, most interviewees selected option c or d: slightly under 66% of the teachers for option c and slightly over 31% for option d. The interviewees who selected option d emphasized that all integer roots are real roots and, thus, this option is more general and clearer than option c. The interviewees who selected option c felt that this approach would not work well if roots are not integers, as indicated in such comments as, "This way is not good if roots are fractions," and "If the root is *(3+√2)/2*…" Real roots included in option d led to the teachers who selected option c both removing option d in their decision-making process and validating option c as the more reasonable assessment. In assessing student's unusual reasoning provided in item 16, the interviewees focused on the meanings of an integer root and real root in the student's reasoning, and the other interviewees

Item number 13

In a unit on simplifying expressions, one of Mr. Serrano's students wrote the following correct solution.

$$\frac{4(a+2)}{3a}+2-\frac{8}{3a}-\frac{6a-1}{6}$$

$$=\frac{4a+8}{3a}+2-\frac{8}{3a}-\frac{6a}{6}+\frac{1}{6}$$

$$=\frac{4a}{3a}+1\frac{6}{6}-a+\frac{1}{6}$$

$$=1\frac{1}{3}+1\frac{7}{6}-a$$

$$=1\frac{2}{6}+1\frac{7}{6}-a$$

$$=2\frac{9}{6}-a$$

$$=3\frac{3}{6}-a$$

$$=3\frac{1}{2}-a$$

Of the following descriptions, which best characterize the student's work?

ⓐ The student knows how to simplify expressions very well and demonstrates strategic use of standard procedures.
ⓑ The student shows good computational skill but does not use processes efficiently.
ⓒ The student knows how to simplify expressions very well, but in the solution the student should write all steps involved in the calculation, such as the step $\frac{8}{3a}-\frac{8}{3a}$.
ⓓ The student should apply a more formal procedure by first finding the common denominator and then adding all terms.

Item number 16

Ms. Quinn asked her students to solve the following quadratic equation.

$$3x^2-6x-24=0$$

Maurice explained, "I added 24 to both sides and divided by 3."

$$3x^2-6x=24$$
$$x^2-2x=8$$
$$x(x-2)=8$$

He then concluded, "The only numbers that are 2 apart and multiply to be 8 are 2 and 4, and -2 and -4, so x has to be 4 or -2." Students agreed that 4 and -2 work when you substitute them into the original equation, but they were unsure about his method.

Of the following statements, which best characterize Maurice's approach to this problem?

ⓐ Maurice's method is wrong because you cannot solve an equation by factoring unless one side of the equation is equal to zero.
ⓑ Maurice's method is wrong because he should have first divided by 3 and then factored the left side of the equation.
ⓒ Maurice's reasoning is correct, but his method often leads to an equation that cannot be solved by inspection.
ⓓ Maurice's reasoning is correct, but his method only works for equations with real roots.

Figure 1. Items 13 and 16 in the MET algebra I assessment (Phelps et al., 2014, p.82 & 84)

started to probe the student's reasoning and then evaluate it among the various ways of solving quadratic equations. The mathematical sensitivity of the terms also helped the teachers' decisions.

Item 13 also includes the student's non-standard reasoning about simplifying an expression and asks teachers to assess it like item 16, as shown in *Figure 1*. The student's reasoning arrives at the mathematically correct answer, and there is no mathematical error. The reasoning involves changing natural numbers into mixed numbers in the process of simplifying expressions, which made this student's work a non-standard solution. Mathematical analysis about evaluating student's reasoning in this case was elusive and difficult. For option a, some interviewees agreed with "The student knows how to simplify expressions very well" because the student got a correct response. Some interviewees, however, disagreed with it, such as "It would be incorrect to conclude that he does not know how to simplify expressions. But it is hard to determine that he knows it very well. It is pretty inefficient to change 2 into 1 6/6." Some interviewees agreed with "strategic use of standard procedures" because, again, the student got a correct answer. Some interviewees disagreed it and evaluated his solution as non-standard and non-strategic reasoning. For option b, some interviewees agreed with the statement that "the student shows good computational skill" because there is no computational error. However, some interviewees disagreed because changing natural numbers into mixed numbers cannot confirm that the student has enough capability to simplify expressions. Even the interviewees who selected the other options agreed that "the student does not use processes efficiently." For option c, some interviewees agreed with the statement that "the student should write all steps," but some interviewees concluded that the student did not have to write down all steps. Some interviewees also commented that the student sometimes wrote down useless steps. The interviewee who selected option d said "To add fractions, I generally teach students to change all fractions into improper fractions, reduce them to a common denominator, and then add them. So, changing 1 into the mixed number in this solution looks unusual … I would teach the formal procedure to this student." Some teachers stated that the steps do not have to be a formal procedure.

Unlike item 16, each option of item 13 does not require a detailed understanding about mathematical language nor advanced mathematics. Furthermore, the interviewees often said they have not seen this kind of student's reasoning in simplifying expressions. Interviewees' responses, for example, included the following: "This reasoning is not typical in my lesson." "Is it (this reasoning) typical in Korea? I have no idea about it in other countries, though." and "Changing the mixed number is more difficult than using improper fractions." Even the teachers who selected the correct answer were not confident about their decisions in the interviews, unlike the case with other items, and spoke about the difficulties reaching decisions for this item. In fact, this item describes a situation where the student got the correct answer, but the reasoning used was not mathematically efficient. Evaluating the student's reasoning in such a situation requires mathematical analysis and decision. Teaching experience could be useless if teachers have not carefully analyzed and assessed this situation. The use of detailed language to specify features of a student's reasoning is also relevant in this item. Because there is no computational error in any of the steps, this reasoning proves the student's good computational capability. However, the steps are not efficient, and, thus, it is hard to say "this student knows how to simplify expressions very well." Thoughtful detailed observation and mathematical assessment of the student's reasoning is imperative in this item.

Second, how to teach is a formidable force in evaluating students' reasoning. In the interviews for item 12, which is shown in *Figure 2*, most teachers were aware of students' typical irrational reasoning of drawing line segments for a curved graph. One interviewee said, for example, "when students draw a graph, they connect points by drawing line segments because they do not understand the graph should be a curve," Another interviewee said that "many students draw the graph like this at the first time," and one other stated that "middle school students often draw line segments between points." And then, to evaluate student explanations, the interviewees, particularly those who selected option d, depended on how they teach students drawing $y=1/x$. "The problem is that they don't have enough points," said one interviewee, for example, "They need to include more points to make it look correct. … I teach it in this way." In fact, this option includes interpolation—the computation of points or values between ones that are known or tabulated using the surrounding points or values. No teachers who selected this option mentioned or justified why interpolation works well mathematically as a base principle. However, interpolation does not justify making line segments to draw the graph because the line segment is still drawn between two points. Even if more points make the graph of *1/x* look correct, it is not correct. The four interviewees thought about being differentiable if derivative of a function exists at a point, and focused on the terminology that students can use or say in the provided situation. They concentrated on what is mathematically founded and what level of understanding students would have in the item. They also mentioned that they offer such comments in their teaching, but they did not depend on their teaching method for their decisions for this item.

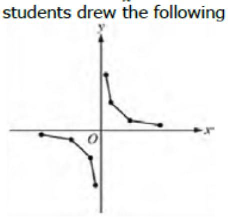
It is challenging to understand and interpret students' talk because it sometimes includes unnecessary comments, unclear reasoning and may be incomplete. Therefore, teachers need to seize what is a mathematical point in student's talk and then evaluate his or her reasoning. This recognition requires epistemological insight and awareness concerning teaching practices. In the interview data for item 17, which is shown in *Figure 2*, the teachers who paid attention to the final sentence in option c, ("We are looking to see what x -values we have to put in $(x-3)^2$ to make the outputs the same as in x^2") and elaborated it mathematically reached a correct answer. However, the interviewees who focused instead on the first two sentences in option c, ("We decided to look at the roots of $y=x^2$ and $y=(x-3)^2$. The vertex of $y=(x-3)^2$ is *(3,0)*, and the vertex of $y=x^2$ is *(0,0)*") determined that these two sentences are inappropriate and then did not try to analyze the last sentence or did not seem to understand it. Features of students' talk in the mathematics classroom are very different from features of rigorous mathematical arguments which unequivocally demonstrate the truth of a given proposition. Because students are involved in the pursuit of learning mathematics, their representations and reasoning could be mathematically flawed, imperfect, inaccurate, or partial. Thus, ways of evaluating or selecting students' comments should be different from one of the

Item number 12

Mr. Jakobsen's students were graphing the function below, where y is inversely proportional to x.

$$y = \frac{1}{x}$$

One of his students drew the following graph.



Mr. Jakobsen has noticed that students often draw graphs with line segments like this despite frequent reminders that the graph should be curved. To get his students to discuss this issue, he asked the class what was wrong with the drawing. Of the following student explanations, which provide the best mathematical explanation of why drawing connected line segments is inappropriate for this graph?

ⓐ When the x changes, the graph should change at the same rate all the time and it shouldn't have corners.

ⓑ The graph changes all the time, but it cannot have sudden changes at some of the points.

ⓒ For any whole number, $\frac{1}{x}$ will always be a rational number and that makes it hard to draw the graph for irrational numbers.

ⓓ The problem is that you don't have enough points. You need to include more points to make it look correct.

Item number 17

In the last class, Mr. Rosen's students graphed quadratics of the form $y = x^2 + c$ for various values of c and developed a rule about shifting the graph of $y = x^2$ up or down. Today Mr. Rosen asked the students to predict what would happen they graphed $y = (x-3)^2$ and then to graph it. Students were surprised that the graph shifted 3 units to the right rather than left or down as they had predicted. Mr. Rosen asked them to explore a little further in groups.

As he walked around the classroom, each group explained to him the strategy they were using to explore the problem. Of the following student descriptions of a strategy for exploring the problem, which is most directly related to the underlying mathematical reason for the graph's behavior?

ⓐ We are trying to prove the rule, so each of us is graphing another one, $y = (x-2)^2$, $y = (x-5)^2$, $y = (x+2)^2$, and $y = (x+1)^2$, and then we will compare our results.

ⓑ We are making a table like yesterday and putting x and x^2 and $(x-3)^2$, so we can plug in different inputs and compare what the outputs are in x^2 and $(x-3)^2$.

x	x^2	$(x-3)^2$
0		
1		

ⓒ We decided to look at the roots of $y = x^2$ and $y = (x-3)^2$. The vertex of $y = (x-3)^2$ is $(3,0)$, and $y = x^2$ is $(0,0)$. We are looking to see what x-values we have to put in $(x-3)^2$ to make the outputs the same as in x^2.

ⓓ We are going to FOIL it out so it will look more like the ones from yesterday, and then we can graph it and compare the way we did yesterday.

Figure 2. Items number 12 and 17 in the MET algebra I assessment (Phelps et al., 2014, p.82 & 84)

mathematician's comments.

The interviewees often depended on how they themselves taught that content in order to evaluate students' reasoning. For example, some of the teachers said "this approach can show diverse cases" for option a in item 17, and "Um… I always use such a way in my class" for option b. They selected their responses based on their teaching experience and then explained why the others are not appropriate. Unlike the teachers who tried to investigate the mathematical points of the provided situation, the teachers who are dependent on their teaching methods are not to focus their attention on the students' reasoning shown in each option. Teachers' teaching methods are based on their own mathematical decisions. However, some teachers in the interviews appeared to have difficulty mathematically uncovering why they teach mathematics in the particular way they do. In other words, adherence to one particular teaching method, which is mathematically vulnerable, hinders teachers' analysis and evaluation students' reasoning. Through an investigation on the teaching methods which are often followed, specialized opportunities should be offered to teachers to explore mathematics precisely and pedagogically.

The third key feature in the teacher interviews is that having a relational understanding regarding the span of mathematics is critical in the context of teaching. For example, the number line is the most popular representation to demonstrate number and operation. The number line is based on the density of rational and irrational numbers. While such knowledge is not exactly used to explain *-2.5 + 3* on the number line to middle school students, all rational and irrational numbers are on the number line. Such knowledge is referred to as horizon content knowledge, which is one of the domains for MKT. Horizon knowledge is "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball, et al., 2008, p.403). The interviews about item 9 and item 10, which are shown in *Figure 3*, illustrate the role of horizon content knowledge in the context of teaching. These two items are about evaluating student reasoning. Item 10 uses a geoboard as a manipulative that students used. Geoboards are not popular in secondary mathematics classrooms in South Korea, and all teachers who were interviewees in this study had no experience using them in their teaching. However, all interviewees recognized that a geoboard is a manipulative which is similar to either whole number points on the coordinate plane or the coordinate plane shown in the GeoGebra software. Moreover, options of item 10 involve the mathematics that students will learn later, such as interpretation of the derivative, density of numbers on the real line, and the parallel postulate. Without an understanding of each option, it is hard to respond to the item, and, therefore, item 10 requires evaluating students' discussion according to these mathematics concepts. Except for one interviewee, all interviewees could clearly explain what each option means and demonstrate which option is related to the students' reasoning. As shown in *Table 1*, the rate of correct responses for this item is 0.742, which is fairly high.

Having horizon content knowledge also helps the teachers select their answers to the item. Option d in item 9, shown in *Figure 3*, contains "calculus." This item involves a student's non-standard reasoning, which is often hard for teachers to analyze and evaluate. However, the interviewees used calculus to exclude option d. Most interviewees reco-gnized that the student's reasoning is mathe-matically correct and, thus, only two options were left to determine the correctness. Then, because the response is not obviously related to calculus, the teachers selected the other option, which is the correct answer. The rate of correct responses for this item of the teachers is 0.883, which is very high, as shown in *Table 1*. Thus, having horizon content knowledge plays a critical role in making mathematical and pedagogical decisions for the context of teaching. The interviewees have a good understanding of those mathematical concepts that are more advanced, generally learned in high school or college, and are connected to one another.

DISCUSSION AND CONCLUSION

This study aims to probe Korean teachers' MKT about algebraic reasoning. The study analyzed the data from 137 teachers' responses to the assessment of MKT for teaching algebra I developed by the MET project and the data from thirteen teachers' interviews. The teachers generally performed well on the assessment, but were relatively lacking in MKT related to function. Also, teaching experience is not statistically related to the level of MKT. Furthermore, mathe-matical analysis is significant in assessing students' reasoning, and each teacher's typical teaching method powerfully influences the teacher's evaluation of students' reasoning. Having horizon content knowledge is a critical component in making decisions in the context of teaching.

Item number 9

Having taught her students to factor quadratics with integer coefficients, integer roots, and a leading coefficient of 1, Ms. Quezada explained that she was going to give them a harder problem. She then asked them to solve the following.

$$3x^2 - 3x - 6 = 0$$

After a few minutes of work, the class discussed their solutions. Letitia said that x was -1 or 2 and explained, "I added $3x$ to both sides and divided by 3."

$$3x^2 - 6 = 3x$$
$$x^2 - 2 = x$$

She then continued "The parabola's just down a little and the line's at 45 degrees, so it's just below zero and about 2 to the right. x can be -1 and 2, and those are the only possible ones."

Of the following, which best characterizes Letitia's approach to this problem?

ⓐ Letitia's <u>method</u> is wrong because she should have first divided by 3 and then factored the left side of the equation.

ⓑ Letitia's <u>method</u> is wrong because this is a parabola and you could graph it, but you would have to graph the original equation and look for the roots.

ⓒ Letitia's <u>reasoning</u> is correct, but her method often leads to points of intersection that might be hard to determine visually.

ⓓ Letitia's <u>reasoning</u> is correct, but her method requires knowledge of calculus.

Item number 10

Ms. Lang's class had been studying the concept of slopes of lines, so she asked them to consider all of the lines passing through one point and how the slops of those lines vary. The students had used geoboards in some earlier work, so they started talking about the slopes of lines on an "infinitely extended" geoboard. (Geoboards are flat blocks of wood, roughly one foot square, with pegs laid out on a grid where rubber bands can be hooked to make lines or polygons.) The students decided that the pegs of the infinite geoboard could be thought of as the set of points with integer coordinates in the Cartesian plane.

During the discussion, students had the following exchange.

Yonah: On the geoboard, you can't get all of the slopes, because the geoboard points are too spread out—there are a whole bundh of lines between the ones you can make.

Andy: I disagree. I think we can make any slope. Starting at one point, by choosing another geoboard point far enough away, we can tilt the line as much or as little as we like

Becky: What I was thinking was if you run a line through one geoboard point, it will always hit another one far enough out.

Of the following concepts, which is most directly related to the mathematics underlying this discussion?

ⓐ Interpretation of the derivative—the derivative is the slope of the tangent line.

ⓑ Density of numbers on the real line—the rational numbers are dense, but not every real number is rational.

ⓒ The parallel postulate—given a point and a line, there is a unique line through the given point parallel to the given line.

ⓓ Each of these concepts is equally related to the underlying mathematics.

Figure 3. Items 9 and 10 in the MET algebra I assessment (Phelps et al., 2014, p.80)

Based on these findings, the current study suggests that professional development should provide teachers with opportunities to mathematically investigate their experiences in instruction, such as their teaching methods or monitoring students' work, and examine their relevance. In the interviews, the teachers often depended on their own teaching experiences in order to analyze mathematical topics and students' work and explanations provided in each item and then make decisions. Such experiences, in fact, do not appropriate decisions. For example, the interviewees spoke about the difficulties reaching decisions for item 13 because they have not observed this kind of non-standard response, but most interviewees' responses to item 12 mentioned their experiences moni-toring students' line segments. Paradoxically, their experiences were not ultimately helpful. The rates of correct responses for item 12 and item 13 are 0.138 and 0.605, respectively, as shown in *Table 1*. Furthermore, teachers' teaching methods frequently operate as the standard for the judgment. Some interviews with item 12 and item 17 show the nature of the teachers' logic, such as the very brief explanation of one teacher that "because I teach it this way, this student's way is appropriate." Although item 16 and the interviewers did not ask interviewees how to teach a quadratic equation, some interviewees spontaneously spoke about how they teach it and used that reference as a criterion to make decisions for this item. This issue of teachers' experience was address by Tirosh, Even and Robinson (1998), which was mentioned in the literature review section. They compared preservice teachers and inservice teachers about their understanding of students' conceptions about expressions and reported that inservice teachers' understanding was better than that of preservice teachers. However, the current study suggests that inservice teachers do not always make appropriate determinations. Postelnicu (2011) also reported inservice teachers' recognition of students' difficulties and work. Yet the current study shows that although teachers have such recognition, it does not necessarily lead to appropriate decisions in the context of teaching mathematics.

Experience is a way to have and accumulate knowledge or skill, but the findings of this study show that experience, teachers' typical teaching methods more specifically, can actually be an obstacle in making relevant decisions in the mathematics teaching situations. Therefore, professional development needs to uncover typical teaching methods adhered to by teachers and probe whether they are mathematically profound and reasonable. The interviews with teachers about item 12 show that there are teachers who teach how to draw a graph with adding more points, and the teacher interviews about item 17 show that some teachers use examples which are not important to capture meanings or understanding of mathematics. Such empirical probes would offer learning opportunities about mathematical knowledge, teachers' practice and students' learning that McCrory and her colleagues (2012) explored. Furthermore, this probe can help teachers reconsider their current practices and examine others as well to learn more about both mathematics and the students they teach, as Ball and Cohen (1999) suggested. In other words, professional development needs to teach MKT as connected and continuous reasoning rather than as separate and fragmented reasoning. Because teaching is constant and spontaneous, reasoning in the context of teaching mathematics should not be taught separately. Teachers also need to have more learning opportunities to interpret and evaluate students' different mathematical explanations and make decisions on what to do in such teaching situations. In terms of offering opportunities to learn to teach mathematics, teacher education in South Korea provides more education on the understanding of theories and philosophy about mathematics education rather than improving MKT (Kim, 2014). There are several teacher education programs and assessments which focus on skills and knowledge that teachers need in teaching situations, such as edTPA (Sato, 2014), annotated lesson plans (Hiebert & Morris, 2012), or teacher education programs focused on high-leverage practices (Ball, Sleep, Boerst & Bass, 2009).

One further point to examine their experiences in instruction is mathematical analysis. Korean teachers have higher advanced mathematical knowledge (Zazkis & Leikin, 2010). Moreover, the current study's findings from the teacher interviews on items 9 and 10, reports that having an intense and fundamental understanding of the span of the mathematics is critically important in the context of teaching. The rates of correct responses for items 9 and 10 are, respectively, 0.883 and 0.729, as shown in Table 1, which are quite high. However, the interviews with item 13 and 16 show a predicament of mathematical analysis of students' work. The interviews with item 13 indicate the difficulties of identifying what makes student's work non-standard and evaluating it appropriately. The interviews with item 16 illustrate the importance of both having sensitivity about mathematical terminology and considering various approaches simultaneously and synthetically to evaluate one of them. In other words, teachers need to have more learning opportunities to interpret and evaluate students' mathematical explanations, as Stump (1999), Asquith, Stephens, Knuth & Alibali (2007), and Lynch and Star (2014) claimed. Furthermore, because this research found that the teachers are relatively weak in functions in the assessment, professional development in South Korea needs to offer more opportunities for teachers to explore and learn functions concerning MKT, primarily flexibility in representations of function and relations between the shift of function and the trans-formation of algebraic representation.

The current study contributes to identifying what needs to be focused on in the curriculum to teach MKT to Korean secondary teachers. Developing a curriculum with practical resources and insight to enact it would be the next step to support Korean secondary teachers in order to support Korean secondary teachers' acquisition of mathematical and pedagogical reasoning that works appro-priately in teaching algebra.

Endnotes

There was a significant difference in the rates of Korean teachers' correct answers for all items (M=0.80, SD=0.21) and those of U.S. teachers (M=0.60, SD=0.25); $t(28)$=3.25, p=0.001. Analysis of the rates of correct answers for U.S. teachers are based on the data from Phelps et al. (2014).

ACKNOWLEDGMENTS

Earlier versions of this paper were presented at the annual meeting of the American Educational Research Association, New York, NY, 2018 and at a conference of 2016 Korea Society of Educational Studies in Mathematics. This paper is part of broader research on mathematical knowledge for teaching, conducted in collaboration with Soo Jin Lee and Inah Ko, whom I wish to thank for their help in initiating this research.

References

American Educational Research Association, American Psychological Association, & National Council on Measurement in Education (1999). *Standards for educational and psychological testing*. Washington, DC: American Educational Research Association.

Asquith, P., Stephens, A. C., Knuth, E. J., & Alibali, M. W. (2007). Middle school mathematics teachers' knowledge of students' understanding of core algebraic concepts: Equal sign and variable. *Mathematical Thinking and Learning*, *9*(3), 249-272.

Attorps, I. (2003). Teachers' images of the 'equation' concept. *European Research in Mathematics Education*, *3*, 1-8.

Ball, D. L., & Cohen, D. K. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In G. Sykes & L. Darling-Hammond (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 3-32). San Francisco: Jossey-Bass.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389-407.

Ball, D. L., Sleep, L., Boerst, T. A., & Bass, H. (2009). Combining the development of practice and the practice of development in teacher education. *The Elementary School Journal*, *109*(5), 458-474.

Belfort, E., Guimaraes, L., & Barbastefano, R. (2001). Tertiary algebra and secondary classroom practices in number and algebra: Closing the gap. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future of the teaching and learning of algebra(Proceedings of the 12th ICMI Study Conference*, (pp. 79-86). Melbourne, Australia: The University of Melbourne.

Blömeke, S., & Paine, L. W. (2008). Getting the fish out of the water: Considering benefits and problems of doing research on teacher education at an international level. *Teaching and Teacher Education*, *24*(8), 2027-2037.

Caglayan, G. (2013). Prospective mathematics teachers' sense making of polynomial multiplication and factorization modeled with algebra tiles. *Journal of Mathematics Teacher Education*, *16*(5), 349-378.

Cai, J., & Knuth, E. (Eds.). (2011). *Early algebraization: A global dialogue from multiple perspectives*. Heidelberg, Germany: Springer.

Chazan, D. (1999). On teachers' mathematical knowledge and student exploration: A personal story about teaching a technologically supported approach to school algebra. *International Journal of Computers for Mathematical Learning*, *4*(2–3), 121-149.

Chazan, D. (2000). *Beyond formulas in mathematics and teaching: Dynamics of the high school algebra classroom*. New York: Teachers College Press.

Chazan, D., Larriva, C., & Sandow, D. (1999). What kind of mathematical knowledge supports teaching for "conceptual understanding"? Preservice teachers and the solving of equations. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 193-200). Haifa, Israel: Technion Printing Center.

Chazan, D., Yerushalmy, M., & Leikin, R. (2008). An analytic conception of equation and teachers' views of school algebra. *The Journal of Mathematical Behavio*r, *27*(2), 87-100.

Chinnappan, M., & Thomas, M. (2001). Prospective teachers' perspectives on function representations. In J. Bobis, B. Perry, & M. Mitchelmore (Eds.), *Numeracy and beyond. Proceedings of the 24th annual conference of the Mathematics Education Research Group of Australasia* (pp. 155-162). Sydney: MERGA.

Cohen, D. K. (2011). *Teaching and its predicaments*. Cambridge, MA: Harvard University Press.

Cole, Y. (2012). Assessing elemental validity: the transfer and use of mathematical knowledge for teaching measures in Ghana. *ZDM*, *44*(3), 415-426.

Delaney, S., Ball, D. L., Hill, H. C., Schilling, S. G., & Zopf, D. (2008). "Mathematical knowledge for teaching": Adapting U.S. measures for use in Ireland. *Journal of Mathematics Teacher Education*, *11*(3), 171-197.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education*, *24*, 94-116.

Haimes, D. H. (1996). The implementation of a "function" approach to introductory algebra: A case study of teacher cognitions, teacher actions, and the intended curriculum. *Journal for Research in Mathematics Education*, *27*(5), 582-602.

Heid, M. K., Blume, G. W., Zbiek, R. M., & Edwards, B. S. (1999). Factors that influence teachers learning to do interviews to understand students' mathematical understandings. *Educational Studies in Mathematics*, *37*(3), 223–249.

Hiebert, J., & Morris, A. K. (2012). Teaching, rather than teachers, as a path toward improving classroom instruction. *Journal of Teacher Education*, *63*(2), 92-102.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, *42*(2), 371-406.

Hill, H. C., Umland, K., Litke, E., & Kapitula, L. R. (2012). Teacher quality and quality teaching: Examining the relationship of a teacher assessment to practice. *American Journal of Education*, *118*(4), 489-519.

Hitt, F. (1994). Teachers' difficulties with the construction of continuous and discontinuous functions. *Focus on Learning Problems in Mathematics*, *16*(4), 10–20.

Huang, R., & Kulm, G. (2012). Prospective middle grade mathematics teachers' knowledge of algebra for teaching. *The Journal of Mathematical Behavior*, *31*(4), 417-430.

Izsák, A., Çağlayan, G., & Olive, J. (2009). Meta-representation in an algebra I classroom. *The Journal of the Learning Sciences*, *18*(4), 549-587.

Kim, Y. (2014). A comparison study of curricular of teacher education for elementary teachers in South Korea and the United States: Focusing on opportunities to learn teaching mathematics. *Journal of Educational Research in Mathematics*, *24*(4), 555-572.

Kim, Y. (2016). Interview prompts to uncover mathematical knowledge for teaching: focus on providing written feedback. *The Mathematics Enthusiast*, *13*(1), 71-92.

Koellner, K., Jacobs, J., Borko, H., Schneider, C., Pittman, M. E., Eiteljorg, E., . . . Frykholm, J. (2007). The problem-solving cycle: A model to support the development of teachers' professional knowledge. *Mathematical Thinking and Learning*, *9*(3), 273-303.

Lewis, J., & Blunk, M. (2012). Reading between the lines: Teaching linear algebra. *Journal of Curriculum Studies*, *44*(4), 515-536.

Lloyd, G. M., & Wilson, M. (1998). Supporting innovation: The impact of a teacher's conceptions of functions on his implementation of a reform curriculum. *Journal for Research in Mathematics Education*, *29*(3), 248-274.

Lynch, K., & Star, J. R. (2014). Teachers' views about multiple strategies in middle and high school mathematics. *Mathematical Thinking and Learning*, *16*(2), 85-108.

McCrory, R., Floden, R., Ferrini-Mundy, J., Reckase, M. D., & Senk, S. L. (2012). Knowledge of algebra for teaching: A framework of knowledge and practices. *Journal for Research in Mathematics Education*, *43*(5), 584-615.

Moses, R. P., & Cobb, C. E. (2001). *Math literacy and civil rights*. Boston: Beacon Press.

Mosvold, R., & Fauskanger, J. (2009). *Challenges of translating and adapting the MKT measures for Norway*. Paper presented at the American Educational Research Annual Meeting in San Diego, CA.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. Washington, DC: US Department of Education.

Ng, D. (2012). Using the MKT measures to reveal Indonesian teachers' mathematical knowledge: challenges and potentials. *ZDM*, *44*(3), 401-413.

Norman, A. (1992). Teachers' mathematical knowledge of the concept of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (Vol. 25, MAA Notes, pp. 215-232).

Washington, DC: Mathematical Association of America.

Phelps, G., Weren, B., Croft, A., & Gitomer, D. (2014). *Developing content knowledge for teaching assessments for the Measures of Effective Teaching study (ETS Research Report No. RR-14-33)*. Princeton, NJ: Educational Testing Service.

Postelnicu, V. (2011). *Student difficulties with linearity and linear functions and teachers' understanding of student difficulties*. Unpublished doctoral dissertation, Arizona State University, Tempe.

Rockoff, J. E., Jacob, B. A., Kane, T. J., & Staiger, D. O. (2011). Can you recognize an effective teacher when you recruit one? *Education Finance and Policy*, *6*(1), 43-74.

Sato, M. (2014). What is the underlying conception of teaching of the edTPA? *Journal of Teacher Education*, *65*(5), 421-434.

Schmidt, W. H., Houang, R. T., Cogan, L., Blömeke, S., Tatto, M. T., Hsieh, F. J., ...Paine, L. (2008). Opportunity to learn in the preparation of mathematics teachers: its structure and how it varies across six countries. *ZDM, 40*(5), 735-747.

Sherin, M. G. (2002). When teaching becomes learning. *Cognition and Instruction*, *20*(2), 119-150.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Stein, M. K., Baxter, J., & Leinhardt, G. (1990). Subject-matter knowledge and elementary instruction: A case from functions and graphing. *American Educational Research Journal*, *27*(4), 639-663.

Strauss, A. L., & Corbin, J. M. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. Thousand Oaks, CA: Sage Publications.

Stump, S. (1999). Secondary mathematics teachers' knowledge of slope. *Mathematics Education Research Journal*, *11*(2), 124-144.

Stump, S. (2001). Developing preservice teachers' pedagogical content knowledge of slope. *Journal of Mathematical Behavior*, *20*, 207-227.

Tanisli, D., & Kose, N. Y. (2013). Pre-Service Mathematics Teachers' Knowledge of Students about the Algebraic Concepts. *Australian Journal of Teacher Education*, *38*(2), 1-18.

Tirosh, D., Even, R., & Robinson, N. (1998). Simplifying algebraic expressions: Teacher awareness and teaching approaches. *Educational Studies in Mathematics*, *35*(1), 51-64.

Wilson, M. R. (1994). One preservice secondary teacher's understanding of function: The impact of a course integrating mathematical content and pedagogy. *Journal for Research in Mathematics Education*, *25*(4), 346-370.

Zazkis, R., & Leikin, R. (2010). Advanced mathematical knowledge in teaching: perceptions of secondary mathematics teachers. *Mathematical Thinking and Learning*, *12*, 263-281.