Journal of Educational Research in Mathematics
2020, Special Issue, 213 ~ 227

An Analysis of Fifth and Sixth Graders' Algebraic Thinking about Reverse Fraction Problems

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ABSTRACT

In light of a growing emphasis on the relationship between fractional knowledge and algebraic thinking, the purpose of this study was to compare algebraic thinking about reverse fraction problems between fifth graders who had not been taught fraction division and sixth graders who had been taught fraction division. For this purpose, we conducted cognitive interviews with 30 students (15 fifth graders and 15 sixth graders) who used different solution methods in a paper-and-pencil test dealing with reverse fraction problems. We analyzed their algebraic thinking in terms of problem structure reasoning, generalization-representation, and justification. The results showed that fifth graders could generalize their solution methods but sixth graders who learned only the algorithm of fraction division had difficulties with generalization and justification. However, sixth graders who reasoned about the problem structure demonstrated proficiency in algebraic thinking. Although some students used the same solution methods in the paper-and-pencil test, they had different capacities in generalizing, representing, and justifying them during the cognitive interviews. The results of this study provide implications for teaching fraction operations in a way that develops algebraic thinking by attending to the mathematical structure and relation.

Key Words: Reverse fraction, Algebraic thinking, Reasoning the problem structure, Generalization, Representation, Justification

INTRODUCTION

As the importance of algebraic thinking has been emphasized in elementary mathematics (e.g., Blanton, Levi, Crites, & Dougherty, 2011; Kieran, Pang, Schifter, & Ng, 2016), interest has grown in the relationship between fractional knowledge and algebraic thinking (Empson, Levi, & Carpenter, 2011; Lee, 2019; Pearn & Stephens, 2018). Considering that fractional competence is a prerequisite for understanding algebra and a predictor for success in algebra (Siegler, Duncan, Davis- Kean, Duckworth, Claessens, Engel et al., 2012), it is important to examine how students learn fraction operations and how students' learning is related to algebraic thinking.

Algebraic thinking has not been formally introduced in the Korean elementary mathematics curriculum but it can be developed when students deal with mathematical structure or relation through typical mathematical topics including fraction operations (Pang & J. Kim, 2018). Given this, Pang and Cho (2019a, 2019b) analyzed how fifth graders in Korea (who had not been taught fraction division) and sixth graders (who had been taught fraction division) solved *reverse fraction problems* using a

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Please cite this article as: Pang, J., Cho, S., & Kwon, M. An analysis of fifth and sixth graders' algebraic thinking about reverse fraction problems. *Journal of Educational Research in Mathematics* (2020). https://doi.org/10.29275/jerm.2020.08.sp.1.213

paper-and-pencil test. An example of reverse fraction problems is "This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with. How many counters did I start with?" (Pearn & Stephens, 2018, p. 241). In this example, the partial quantity (i.e., 10) and equivalent fraction of the partial quantity (i.e., $\frac{2}{3}$) are given, whereas the quantity of the whole (i.e., $\frac{2}{3}$) is unknown.

To be clear, the reverse fraction problem can be solved in multiple ways without using the algorithm of fraction division (Pearn & Stephens, 2018). However, applying the algorithm makes it easy for students to solve the problem. The previous studies (i.e., Pang & Cho, 2019a, 2019b) showed how fifth graders and sixth graders solved reverse fraction problems in various contexts. However, the paper-and-pencil test used in the previous studies had limitations in revealing the students' in-depth algebraic thinking. On the one hand, some solution methods used by fifth graders indicated that they understood the problem structure despite some mathematical errors in representing their ideas. On the other hand, other solution methods used by sixth graders made us question whether they understood the meaning of mathematical expressions that they wrote and whether they could justify their generalized methods in their own words. Considering the limitations of the paper-and-pencil test used in the previous studies, we conduct cognitive interviews for this current study, focusing on the algebraic thinking processes (Kaput, 2008).

Given this background, the purpose of this study was to examine similarities and differences in how fifth graders and sixth graders reasoned about the problem structure, represented the generalization with words and variables, and justified the generalization. The research questions were as follows: (a) Among fifth-grade students who were not taught the division of fractions, what algebraic thinking processes did the students use in solving reverse fraction problems? (b) Among sixth- grade students who were taught the division of fractions, what algebraic thinking processes did they use to solve reverse fraction problems compared with those used by the fifth graders? The findings of this study provide implications for teaching elementary students algebraic thinking in conjunction with fractional knowledge.

THEORETICAL BACKGROUND

1. Students' Fractional Knowledge in Solving Reverse Fraction Problems

Building on previous studies (Hackenberg & Lee, 2015; Lee & Hackenberg, 2014), Pearn and Stephens (2018) designed reverse fraction problems and then used a paper-and-pencil test and cognitive interviews to analyze how fifth graders and sixth graders in Australia understood fraction structure and generalized. The term *reverse fraction problem* has not been formally introduced in Korean elementary mathematics textbooks, but, as shown in *Figure 1*, similar problems can be found in the lesson titled "Finding Out Base Quantity Using Rate and Quantity Compared," which is taught during the first semester of Grade 6 (Ministry of Education [MoE], 2018, p. 114). The sixth-grade students in Korea learn the division of fractions before solving the reverse fraction problem. Regarding an equation with the structure (whole number) ÷ (fraction) that is related to the reverse fraction problem, students learn two methods for solving it: First, they learn to calculate it as the division of fractions with the same denominator by converting the whole number (i.e., dividend) into a fraction with the same denominator as the divisor. Second, they learn to multiply the whole number by the reciprocal of the divisor (MoE, 2018)[1].

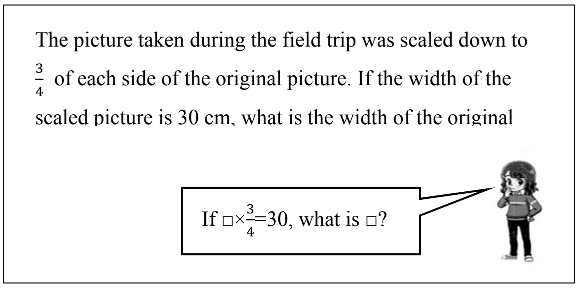


Figure 1. An example related to reverse fraction problems in Korean mathematics textbooks (MoE, 2018, p. 114)

Given this background, two of this current study's authors, Pang and Cho, published previous research (2019a, 2019b) in which we administered a paper-and-pencil test about reverse fraction problems to fifth and sixth graders in Korea and compared the results with

1 Six graders in Korea have used new mathematics textbooks since 2019. The sixth-grade students in this study learned the division of fractions using the previous textbook, which is the textbook that was briefly reviewed here.

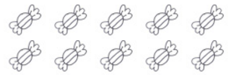
those of Pearn and Stephens (2018). In this process, we (i.e., Pang and Cho, 2019a, 2019b) expanded the reverse fraction problems to include diverse problem contexts and revised Pearn and Stephens' analytical framework. First, we expanded the three reverse fraction problems from Pearn and Stephens' test into 12 problems by considering multiple problem contexts (see *Table 1* for examples). The original three problems were thought to make it difficult for us to examine whether students might persist in using the generalized methods if the problems' numbers or their characteristics were changed (Pearn, Pierce, & Stephens, 2017). We first categorized whether the partial quantity in the problem context was discrete or continuous. If it was continuous, the context of fractions was added. We also considered whether the fraction that represented the partial quantity was less than 1 or greater than 1. As the presentation of diagrams in the problem might influence how elementary school students solved it, we also considered diagrams in the problem contexts. Consequently, we used 12 different problems. For instance, P-1 in *Table 1* (i.e., "Minseo has 10 candies. This is $\frac{2}{3}$ of the number of Jihyun's candies. How many candies does Jihyun have?") was presented with the diagram of 10 candies (see *Table 2* for the diagram). In contrast, P-7 was presented without a diagram as follows: "I have 12 books. This is $\frac{2}{3}$ of the number of Soojin's books. How many books does Soojin have?"

Second, building on Pearn et al.'s (2017) analytical framework (i.e., additive, partially multiplicative, multiplicative, and advanced multiplicative), we further specified the classification and analyzed students' solution methods more systematically, as shown in *Table 2*. Because our students tended to use the diagrams along with another solution method, we specified the solution methods as (a) visual and additive, (b) visual and partially multiplicative, (c) visual and multiplicative, and (d) visual and advanced multiplicative. In this way, we investigated how the presentation of diagrams in the problems helped students' solution methods. Considering that the use of the multiplicative method is a precursor of generalization (Pearn & Stephens, 2018), we categorized both multiplicative and advanced multiplicative methods as high-level solution methods, whereas we distinguished other solution methods as low-level solution methods. If a student solved the problems with multiple methods, we chose the most

Table 1. Examples of reverse fraction problems in Pang and Cho's study (2019a, p. 5)

Problem context				Problem number in the paper-and-pencil test	Examples
Partial quantity		Fraction that represents the partial quantity	Diagrams		
Discrete	Whole number	Less than 1	Yes	P-1	Minseo has 10 candies. This is $\frac{2}{3}$ of the number of Jihyun's candies. How many candies does Jihyun have?
			No	P-7	
		Greater than 1	Yes	P-1	Kyungsu has 21 Baduk-stones now. This is $\frac{7}{6}$ of the number which Kyungsu has at home. How many Baduk-stones does Kyungsu have at home?
			No	P-8	
Continuous	Whole number	Less than 1	Yes	P-3	The length of my pencil is 20cm. This is $\frac{4}{7}$ of the length of Yelim's pencil. What is the length of Yelim's pencil?
			No	P-9	
		Greater than 1	Yes	P-4	My weight is 18kg. This is $\frac{3}{2}$ of Sunghyun's weight. What is the weight of Sunghyun?
			No	P-10	
Continuous	Fraction	Less than 1	Yes	P-5	The weight of Seonyoung's pencil case is $\frac{3}{7}$kg. This is $\frac{3}{5}$ of the weight of my pencil case. What is the weight of my pencil case?
			No	P-11	
		Greater than 1	Yes	P-6	The length of a curtain in a classroom is $\frac{4}{7}$m. This is $\frac{4}{3}$ of the length of a window. What is the length of the window?
			No	P-12	

Table 2. Analytical framework for a paper-and-pencil test in Pang and Cho's study (2019a, p. 6)

P-1) Minseo has 10 candies. This is $\frac{2}{3}$ of the number of Jihyun's candies. How many candies does Jihyun have?	
Solution methods	Illustrations
Advanced multiplicative	Using the division of fractions. e.g.) $\frac{2}{3} \div 2 = \frac{1}{3}$, $\frac{1}{3} \times 3 = \frac{3}{3}$, $10 \div 2 = 5$, $5 \times 3 = 15$; $10 \div \frac{2}{3} = 10 \times \frac{3}{2} = 15$
Multiplicative	Finding the quantity represented as the unit fraction by dividing the partial quantity by the numerator and then multiplying it by the denominator to find the whole. e.g.) $10 \div 2 = 5$, $5 \times 3 = 15$; $\frac{2}{3} = 10$, $\frac{1}{3} = 5$, $5 \times 3 = 15$ Finding the quantity represented by the unit fraction using division and then multiplying the same amount with the numerator and the denominator. e.g.) $\frac{1}{3} = 5$, $2 \times 5 = 10$, $3 \times 5 = 15$
Partially multiplicative	Finding the quantity represented by the unit fraction and then adding or subtracting the quantity to find the number of objects needed to represent the whole. e.g.) $10 \div 2 = 5$, $10 + 5 = 15$; $2 \times 5 = 10$, $10 + 5 = 15$; $\frac{1}{3} = 5$, $\frac{1}{3} + \frac{2}{3} = 15$
Additive	Finding the quantity represented by the unit fraction and then using repeated addition to find the number of objects needed to represent the whole. e.g.) $10 \div 2 = 5$, $5 + 5 + 5 = 15$; $\frac{1}{3} = 5$, $5 + 5 + 5 = 15$; $\frac{1}{3} = 5$, $\frac{2}{3} = 10$, $\frac{3}{3} = 15$
Visual	Using the diagram in the problem or solving the problem by drawing a picture: These cases are classified as visual methods. Using other solution methods along with visual methods: These cases are classified as (a) visual and additive, (b) visual and partially multiplicative, (c) visual and multiplicative, and (d) visual and advanced multiplicative.

frequently used method. For instance, when a student solved six out of 12 problems, five with a partially multiplicative method and one with a multiplicative method, we coded the methods as a partially multiplicative method. If the solution methods were used with equal frequency, we chose the higher-level solution method as the student's method. For instance, when a student solved six out of 12 problems, three of them by a partially multiplicative method and the other three problems by a multiplicative method, we coded them as a multiplicative method.

We found that the fifth graders' most frequently used method was the multiplicative method. This method is similar to the methods defined as *understanding of equivalence and transformation using equivalence* in Pearn and Stephens' study (2018). Fifth graders who were not taught the division of fractions solved the problem by reasoning the structure of the reverse fraction problem. Most sixth graders solved the problems using the advanced multiplicative method because they had already learned fraction division. The correct-answer rate of the sixth graders was relatively high (80%), but it was difficult for us to discern whether they reasoned the problem structure in the paper-and-pencil test.

Because we revised the reverse fraction problems and analytical framework, we could identify the problem contexts with high correct-answer rates and high-level solution methods. The problem contexts that fifth graders easily assessed and the high-level solution methods that they used were different from those of sixth graders. Still, we found some commonalities between fifth and sixth graders. First, when the problems were presented with diagrams, both fifth and sixth graders had higher correct-answer rates and solved the problems in various ways. Unlike the findings of Pearn and Stephens (2018), our findings indicated that students who used the visual method did not simply partition the diagrams or count by grouping. Instead, they visually represented the problem structure and often incorporated high-level solution methods such as a multiplicative method. In addition, sixth graders who had already learned fraction division solved the problems better with the diagrams. These findings support the idea that visual representations such as diagrams help students understand problems and think algebraically (Cooper & Warren, 2011; Moss & McNab, 2011).

We also found a second commonality. Even if the numbers were changed in the reverse fraction problems, both fifth and sixth graders who used high-level solution methods (i.e., multiplicative or advanced multiplicative) tended to solve the problems using the same method. As Pearn and Stephens (2018) suggested, these students used generalizable methods. In other words, students using high-level solution methods could generalize their own methods regardless of the problem contexts. However, our paper-and-pencil test did not provide sufficient evidence to judge whether students using high-level solution methods were thinking more algebraically than those using low-level solution methods.

2. A Framework to Analyze Algebraic Thinking: Problem Structure Reasoning, Generalization-Representation, Justification

To analyze students' algebraic thinking processes about reverse fraction problems, Pearn and Stephens (2018) focused on *understanding of equivalence, transformation using equivalence, and the use of generalizable methods*. They posited that students' algebraic thinking is revealed as long as the students use a multiplicative method based on understanding the relationship between partial quantity and equivalent fraction, solve the problems using the same method when the numbers in the problems are changed, and represent their solution method using algebraic notations. However algebraic thinking is a more complex process. Kaput (2008) specified that algebraic thinking comprised both content strands and thinking processes and defined it as generalizing, representing and justifying the generalization in various ways, as well as reasoning based on the generalization. Building on Kaput (2008), Blanton, Brizuela, Stephens, Knuth, Isler, Gardiner et al. (2018) constructed a conceptual framework for algebraic thinking based on generalizing, representing generalizations, justifying generalizations, and reasoning with generalizations. Thinking algebraically means the ability to reason a mathematical relationship and structure and make a generalization (Kieran et al., 2016).

It is important to attend to the structure to generalize from specific cases (Radford & Roth, 2011). To help students think algebraically, Russell, Shifter, and Bastable (2011) investigated the structure of operations, and Ng (2015) suggested a model to visualize the problem structure. After reasoning the problem structure, students need to find out the pattern of each problem context, represent it in a generalizable way, and justify the generalization. Here, *generalization* is a process of identifying the structure and relation of mathematical context (Blanton et al., 2011). Through teachers' questions — such as "Can you explain the relationship?" or "Does it always work?" — students have an opportunity to generalize the relationship between quantities and the structure of the operation. *Representation of generalization* includes speech, algebraic notation using variables, tables, graphs, gestures, and actual actions (Radford, 2011). Lastly, *justification of generalization* is needed. Justification supports algebraic thinking because it pays attention to the structure that is the basis of calculation (Blanton et al., 2011). If a student solved the given problem well but could not justify it, it is likely that he or she solved the problem just by applying a related procedure or algorithm (Lobato, Ellis, Charles, & Zbiek, 2010). Given this theoretical background, this current study analyzed fifth graders' and sixth graders' algebraic thinking processes in terms of problem structure reasoning, generalization-representation, and justification.

METHODS

1. Participants

The purpose of this study was to examine students' algebraic thinking in solving reverse fraction problems in depth. Among 46 fifth graders[2](Pang & Cho, 2019a) and 133 sixth graders[3](Pang & Cho, 2019b) who completed the paper-and-pencil test, we purposefully selected 15 fifth graders and 15 sixth graders for cognitive interviews. The selection criteria included choosing at least one student per solution method in the paper-and-pencil test. Note that no fifth graders used the advanced multiplicative

2 We conveniently sampled four elementary schools in Seoul and five elementary schools in Chungcheongbukdo. We selected one classroom from each of the nine schools and administered the paper-and-pencil test to students. Among a total of 207 students, we excluded 38 students who had already learned fraction divisions through private lessons and 123 students who solved fewer than four problems. It was unsurprising that such a large proportion of the fifth graders were unable to solve four or more problems since they had not yet had the opportunity to solve reverse fraction problems in their classroom lessons. As a result, we analyzed the solution methods of 46 students.

3 We used the same sampling method for sixth graders. We administered the paper-and-pencil test to 198 sixth graders from nine elementary schools. Among a total, we excluded 38 students who solved fewer than four problems and 27 students who had learned equations through private lessons. As a result, the sample size for the analysis was 133.

method. Only one sixth-grade student used an additive method, but we had difficulties with recruiting this student for cognitive interviews. The rationale for this selection was to figure out whether students who used the low-level solution methods in the paper-and-pencil test could use multiplicative or advanced multiplicative methods in the cognitive interviews. After that, we chose fifth graders who used multiplicative methods and sixth graders who used advanced multiplicative methods because we wanted to look closely at algebraic thinking by students who used the same solution method in the paper-and-pencil test. Students who used multiplicative or advanced multiplicative methods tended to use the same method regardless of the problem context (Pang & Cho, 2019a, 2019b). Pearn and Stephens (2018) argued that using generalizable methods is an indicator of algebraic thinking. *Table 3* summarizes the participants selected for in-depth cognitive interviews.

2. Cognitive Interview Tasks

Using the 12 problems in the paper-and-pencil test (see *Table 1*), we first examined how the students solved reverse fraction problems in which the partial quantity and fraction representing the partial quantity were given. In doing so, we particularly focused on how fifth graders solved the problems since they had not been taught fraction division yet. Also, we focused on how sixth graders reasoned the problem structure.

As illustrated in *Table 4*, we then provided three additional problems in the cognitive interviews in which the partial quantity and/or the fraction representing the partial quantity were unspecified. The purpose of these additional problems was to examine whether students could generalize their solution methods and represent them using variables. Similar to the cognitive interview tasks administered by Pearn and Stephens (2018, p. 252), we did not provide diagrams. We chose this approach because the ambiguity of problems without diagrams may stimulate students' algebraic thinking (Carraher & Schliemann, 2007). Also, the absence of diagrams makes it difficult to use visual or additive methods. We aimed to examine whether students could generalize the solution methods and represent them using variables in a meaningful way.

3. Data Collection and Analysis

During the cognitive interviews, students verbally explained their thinking, wrote down their thinking processes, or used diagrams to solve the problems. Pearn and Stephens (2018) asked students to solve the problems without diagrams, but we analyzed the

Table 3. Solution methods in the paper-and-pencil test of the cognitive interview participants

	Visual*	Additive	Partially multiplicative	Multiplicative	Advanced multiplicative
Year 5 (n = 15)	2	1	1	11	-
Year 6 (n = 15)	1	-	1	2	11

*Visuals: Two fifth-graders used visuals plus the partially multiplicative method, whereas one sixth grader used visuals plus the multiplicative method.

Table 4. Additional tasks in the cognitive interviews

Problem context			Problem number in cognitive interviews	Problem	Note
Partial quantity	Fraction that represents the partial quantity	Diagrams			
Unspecified	Less than 1	No	I-1	Minseo has any number of candies, which is $\frac{2}{3}$ of the number of Jihyun's candies. How can you represent the number of Jihyun's candies?	Revised P-1: the fraction representing the partial quantity is not changed, but the partial quantity is unspecified.
	Greater than 1	No	I-2	My weight is any kg, which is $\frac{3}{2}$ of Sunghyun's weight. How can you represent Sunghyun's weight?	Revised P-4: the fraction representing the partial quantity is not changed, but the partial quantity is unspecified.
	Unspecified	No	I-3	Minseo has any number of candies, which is any fraction of the number of Jihyun's candies. How can you represent the number of Jihyun's candies?	Revised I-1: both the fraction representing the partial quantity and the partial quantity are unspecified.

cases with diagrams. We did so because various visual representations, such as drawings or tables, facilitate algebraic thinking (Lee, 2019). On average, each cognitive interview took about 20 minutes. All interviews were videotaped and transcribed. Transcripts and students' written works were used as data to analyze students' algebraic thinking processes.

If students could generalize the high-level methods and represent the unspecified quantity using variables, Pearn and Stephens (2018) classified the students' processes as algebraic thinking and evaluated the level of algebraic thinking from Level 0 to Level 4. To examine the features of algebraic thinking in depth, we revised the analytical framework by considering the following two aspects.

First, we specified *algebraic thinking* as problem structure reasoning, generalization-representation, and justification. By including justification, we aimed to analyze students' thinking processes in various respects. In terms of generalizing the solution method, we divided it into two cases: verbal representation and variable representation. Because verbal representation is usually addressed earlier than variable representation in Korean mathematics textbooks, we examined whether students could generalize the solution methods in words first and then represent them with variables.

We also analyzed whether the interviewees reasoned the structure of reverse fraction problems. This reasoning is related to the *understanding of equivalence* and *transformation using equivalence* process attributes in Pearn and Stephens (2018), but we divided reasoning into three cases: (a) whether students understood the relationship between the partial quantity and fraction representing the partial quantity; (b) whether students understood the quantity of the whole as $\frac{n}{n} = 1$; and (c) whether students knew that they needed to find the unit fractions first. Furthermore, we examined whether students who seemed to reason the problem structure but used the low-level methods in the paper-and-pencil test could solve the problems using the high-level methods (multiplicative or advanced multiplicative) during cognitive interviews. In terms of generalization-repre- sentation, we analyzed whether students could find similarities among solution methods, generalize and verbally represent, and represent equations using variables. Lastly, we analyzed justification, assessing whether students could logically explain their solution methods and generalization processes.

Second, we focused on students' algebraic thinking processes. To be clear, Pearn and Stephens (2018) classified their interviewees' algebraic thinking levels based on whether they consistently used multiplicative methods and employed suitable algebraic notations. As seen in *Table 5*, however, our analytical framework for cognitive interviews was different, consisting of problem structure reasoning, generalization-representation (verbal), generalization -representation (variables), and justification. Given this framework, we classified each algebraic thinking process into three levels (i.e., high, medium, and low). We considered a student's algebraic thinking level to be *high* if their responses did not have any logical flaws, *medium* if the responses were logically insufficient, and *low* if the responses did not include any information about problem structure reasoning, generalization-rep-resentation, and justification. For example, in the case of generalization-representation (verbal), we scored students *high* when they attended to the structure of reverse fraction problems in verbally representing their generalized solution methods, *medium* when they simply represented the algorithm in words based on memorization and *low* when they could not generalize their solution methods. In this way, we analyzed the features of fifth and sixth graders' algebraic thinking processes with a total of 15 reverse fraction problems (see *Table 1* and *Table 4*).

RESULTS

1. Fifth Graders' Algebraic Thinking Process

Table 6 shows fifth graders' solution methods and algebraic thinking processes. Notably, 14 out of the 15 fifth graders who participated in cognitive interviews solved the problems using the structure of reverse fraction problems, but only one fifth grader demonstrated proficiency in all four aspects of algebraic thinking (see the shaded row in *Table 6*). Also noteworthy is that 11 students who used the same multiplicative method in the paper-and-pencil test revealed different algebraic thinking processes (see the first five rows in *Table 6*). Lastly, the most frequently used algebraic thinking process among the fifth graders (n=4) involved high problem structure reasoning, high generalization-representation (verbal), medium generalization-representation (variables), and high justification (see the bold in *Table 6*). In other words, the fifth graders could solve the reverse fraction problems using the multiplicative method both in the paper-and-pencil test

Table 5. Analytical framework for cognitive interviews

Algebraic thinking process	Focus of analysis
Problem structure reasoning	• Did students solve the reverse fraction problems by reasoning the structure behind them? • Were students able to use the high-level methods compared to the methods they used in the paper-and-pencil test?
Generalization-representation (verbal)	• Did students generalize their solution methods and verbally represent the generalization?
Generalization-representation (variables)	• Did students use variables to represent the unspecified quantity in the problem? • Did students generalize their solution methods and represent them in equations with variables?
Justification	• Did students logically explain their solution methods? • Did students logically explain the generalization process involved in the solution methods?

Table 6. Fifth graders' solution methods and their algebraic thinking processes (n=15)

Solution methods		Algebraic thinking processes				Number of students	Note
Methods used in the paper-and-pencil test	Additional methods used in the cognitive interviews	Problem structure reasoning	Generalization-representation (verbal)	Generalization-representation (variables)	Justification		
Multiplicative	-	high	high	high	high	1	-
Multiplicative	-	high	high	high	medium	2	-
Multiplicative	-	high	high	medium	high	4	See *Episode 2*
Multiplicative	-	high	high	medium	medium	3	-
Multiplicative	-	low	low	low	low	1	-
Partially multiplicative	Multiplicative	high	high	medium	medium	1	-
Visual + partially multiplicative	Multiplicative	high	high	medium	medium	2	-
Additive	Multiplicative	high	high	medium	high	1	See *Episode 1*

and the cognitive interview, reason the problem structure, generalize their solution method, represent and justify it verbally. However, they had difficulties when attempting to represent their solutions with variables. The following sections provide more details of the fifth graders' algebraic thinking processes.

1) Problem Structure Reasoning

As mentioned above, 14 out of the 15 fifth graders reasoned the structure of reverse fraction problems. This finding is consistent with the fifth graders' performance in the paper-and-pencil test (Pang & Cho, 2019a). This result confirmed that the fifth graders could solve the problems that they had not learned yet by reasoning the problem structure. Four fifth graders who did not use the multiplicative method in the paper-and-pencil test solved the problems during their cognitive interviews using the multiplicative method (see the bottom three rows in *Table 6*). These students used different solution methods in the paper-and-pencil test but found the quantity of the unit fraction by reasoning the problem structure. As the interviewer probed them about the quantity of the unit fraction to find out the quantity of the whole, students solved the problems using the high-level multiplicative method (see *Episode 1* for an example). In summary, fifth graders solved the problems by reasoning the problem structure even though they had not yet learned it in their classroom mathematics lessons. They were able to use the multiplicative solution method either in the paper-and-pencil test or in the cognitive interviews.

Episode 1. A fifth grader who used the additive method in the paper-and-pencil test but employed the multiplicative method during the cognitive interview

Interviewer: How did you solve problem number 3? (See P-3 in *Table 1*.)

Student 1: As I wrote here, 20cm is $\frac{4}{7}$ of Yelim's pencil. $\frac{5}{7}$ is 25, $\frac{6}{7}$ is 30, and $\frac{7}{7}$ is 35.

Interviewer: Can you find out $\frac{7}{7}$ directly without using $\frac{5}{7}$ and $\frac{6}{7}$?

Student 1: 20 divided by 4 is 5.5 multiplied by 7 is 35.

Interviewer: You divided 20 by 4 and got 5? What does it equal to?

Student 1: $\frac{1}{7}$. Because I multiplied 7, it's 35. $\frac{7}{7}$ is 35.

2) Generalization-Representation and Justification

Regardless of the solution methods used in the paper-and-pencil test, 14 fifth graders who reasoned the structure of reverse fraction problems could generalize the multiplication method and verbally represent during cognitive interviews. However, the other student who used the multiplicative method in the paper-and-pencil test could not reason the problem structure, which resulted in low performance in the subsequent algebraic thinking processes. This student just multiplied the same amount to the numerator and the denominator and did not generalize the solution method. Reasoning the structure of reverse fraction problems impacts how students find similarities among solution methods and generalize them.

Fourteen out of 15 fifth graders generalized their own solution methods and verbally represented the solution, but they tended to use specific numbers when the quantities were unspecified in the problem. This type of response is consistent with the finding of Pang and J. Kim (2018) that third graders had difficulties with representing unspecified quantities using variables. After the fifth graders in this study represented the unspecified quantities using □, they had difficulties in writing an equation using variables and used the less-elaborated representations. For instance, some students drew a picture representing the equation using variables (see *Figure 2*), whereas others used an equation such as $\square = \frac{3}{2}$ (see *Figure 3*). However, they understood what each variable meant. This approach showed that fifth graders can use variables meaningfully in their generalization processes, even though their representations are not very proficient (Brizuela, Blanton, Sawrey, Newman-Owens, & Gardiner, 2015).

Fourteen fifth graders who logically reasoned the structure of reverse fraction problems could explain their own solution methods and generalization processes. They reasoned the structure of reverse fraction problems and solved them. However, eight students did not fluently justify their thinking processes, and we scored them as medium in terms of justification. These results are understandable when we consider that the fifth graders had not yet learned about reverse fraction problems in classroom lessons. Contrary to this trend, *Episode 2* illustrates the case of a fifth grader who logically justified his solution process, even though he did not represent the equation using variables well (as shown in *Figure 3*).

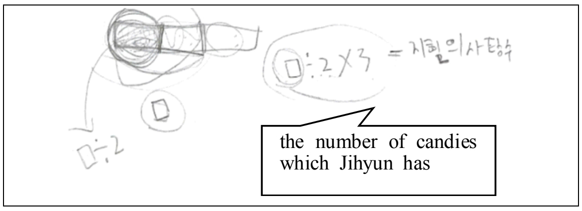


Figure 2. An example of a fifth grader using variables

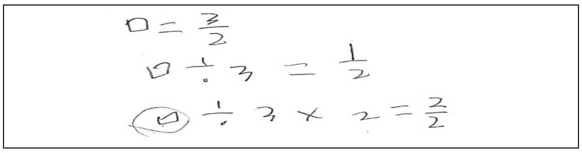


Figure 3. Another example of a fifth grader using variables

Episode 2. An example of a fifth grader who justified his solution method well

Interviewer: Can you explain how you solved problem number 1? (See P-1 in *Table 1*)

Student 2: If it's $\frac{2}{3}$, it's 10 divided by 2. It's 5. 3 is the total, so 5 multiplied by 3 is 15.

Interviewer: Why did you divide 10 by 2?

Student 2: Because it's $\frac{2}{3}$. It's not 1, so I divided it by the denominator, 2. I got 5 and then multiplied it by 3.

Interviewer: Why did you multiply 5 by 3?

Student 2: If $\frac{1}{3}$, 1[the numerator of $\frac{1}{3}$] can be included three times [for $\frac{2}{3}$]

Interviewer: Ah, is the number of Jihyun's candies equal to three times of $\frac{1}{3}$?

Student 2: Yes.

Interviewer: If so, what fraction is the candies that Minseo has?

Student 2: $\frac{2}{3}$

To summarize, during the cognitive interviews, all but one fifth-grade student solved the problems by reasoning the structure of reverse fraction problems. They solved the problems using the multiplicative method and could verbally generalize their solution methods. Even though some of the fifth graders' representations with variables were not mathemati- cally accurate, their overall algebraic thinking pro- cesses were interpreted as meaningful.

2. Sixth Graders' Algebraic Thinking Process

Table 7 shows sixth graders' solution methods and their algebraic thinking processes. The characteristics of the sixth graders' algebraic thinking processes are as follows, in comparison with those of the fifth graders summarized in *Table 6*. First, only seven out of 15 sixth graders solved the problems by reasoning the structure of reverse fraction problems. Considering that 14 fifth graders reasoned the problem structure, seven was a very small number. This result impacted how sixth graders generalized their solution methods, verbally represented, and justified them. More specifically, only seven sixth graders verbally represented the generalized solution method without any difficulties (compared to 14 fifth graders) and five sixth graders justified well (compared to six fifth graders). Second, four sixth graders were excellent (see the shaded rows in *Table 7*) in all aspects of their algebraic thinking processes (compared to only one fifth grader), and 12 sixth graders represented their equations using variables (compared to only three fifth graders). Third, similar to the findings concerning fifth graders, 11 sixth graders who used the advanced multiplicative method demonstrated different algebraic thinking processes (see the first four rows in *Table 7*). Lastly, the type of algebraic thinking process most frequently employed by sixth graders (n= 6) involved low problem structure reasoning, medium generalization-representation (verbal), high generalization-representation (variables), and low justification (see the bold in *Table 7*). In other words, the students solved the reverse fraction problems using the advanced multiplicative method both in the paper-and-pencil test and the cognitive interview and represented the equations accurately using variables, but they had some difficulties when verbally generalizing and justifying the solution method. The following sections provide more details about the features of sixth graders' algebraic thinking processes.

1) Problem Structure Reasoning

In our previous study of sixth graders (Pang & Cho, 2019b), 85% of the sixth graders solved the problems using the advanced multiplicative method. From these findings from the previous study, it was

Table 7. Sixth graders' solution methods and algebraic thinking processes (n = 15)

Solution methods		Algebraic thinking processes				Number of students	Note
Methods used in the paper-and-pencil test	Additional methods used in the cognitive interviews	Problem structure reasoning	Generalization-representation (verbal)	Generalization-representation (variables)	Justification		
Advanced multiplicative	Multiplicative	high	high	high	high	2	
Advanced multiplicative	Multiplicative	high	high	high	medium	2	
Advanced multiplicative	**-**	**low**	**medium**	**high**	**low**	**6**	See *Episode 3*
Advanced multiplicative	-	low	medium	medium	low	1	
Multiplicative	Advanced multiplicative	high	high	high	high	1	See *Episode 4*
Multiplicative	Advanced multiplicative	high	high	medium	high	1	
Partially multiplicative	Multiplicative	low	low	low	low	1	
Visual +multiplicative	Advanced multiplicative	high	high	high	high	1	

difficult for us to know whether the students understood the structure of reverse fraction problems by just looking at their solution processes in the paper-and-pencil test. The cognitive interviews in this current study showed that only seven out of 15 sixth graders reasoned the problem structure. Among these students, four students used the advanced multiplicative method in the paper-and-pencil test and could use the additional multiplicative method during cognitive interviews. Another two students who used the multiplicative method in the paper-and-pencil test and the remaining one who used visuals along with the multiplicative method in the paper-and-pencil test were used the advanced multiplicative method during cognitive interviews.

Eight sixth graders did not reason the problem structure. Among them, seven sixth graders solved the problems using the advanced multiplicative method in the paper-and-pencil test but were unable to use other methods during cognitive interviews. They responded that they could solve the problems using only fraction division and did not understand the structure of reverse fraction problems at all. One student who used a partially multiplicative method in the paper-and-pencil test could use the multiplicative method during the cognitive interview but had difficulties with reasoning the problem structure by simply multiplying the same number with the numerator and denominator.

Because sixth graders had already learned the division of fractions in classroom lessons, they could use high-level solution methods in the paper-and-pencil test (Pang & Cho, 2019b), but the number of sixth graders who reasoned the problem structure was less than that of fifth graders. However, similar to the fifth graders, the sixth graders who reasoned the problem structure were able to use higher-level solution methods during the cognitive interviews than the paper-and-pencil test. Also, they employed other solution methods besides the advanced multiplicative method.

2) Generalization-Representation and Justification

As mentioned earlier, sixth graders represented the equation using variables better than fifth graders. Because the equation using □ is introduced in the lesson of “Finding Out Base Quantity Using Rate and Quantity Compared” in the sixth-grade mathematics textbook (see *Figure 1*), some sixth graders could write an equation using □ to solve the problems in the paper-and-pencil test. For that reason, they did not have difficulties writing an equation with □, as long as they represented the unspecified amount as □. In the third problem presented in the cognitive interviews (see I-3 in *Table 4*), both the partial quantity and the fraction representing the partial quantity were unspecified, so three variables were needed. For this problem, fifth graders had difficulties with writing an equation, but some sixth graders accurately wrote an equation using three different symbols from the beginning (see *Figure 4*). Students who used the same symbol for the three variables realized that they needed to use different symbols and revised their initial equations. As shown in *Figure 5*, sixth graders who used both the advanced multiplicative method and the multiplicative method could represent the equation with variables by generalizing both methods.

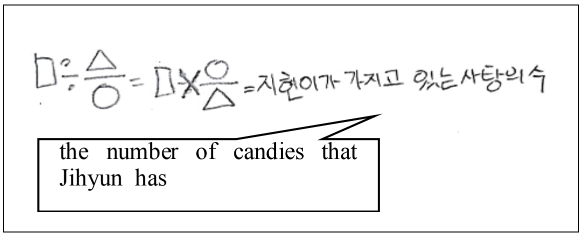


Figure 4. An example of a sixth grader using variables

Notably, six out of the eight sixth graders who did not accurately reason the problem structure could still represent the equation using variables for the generalized solution method. However, as they did not reason the problem structure, they had difficulties

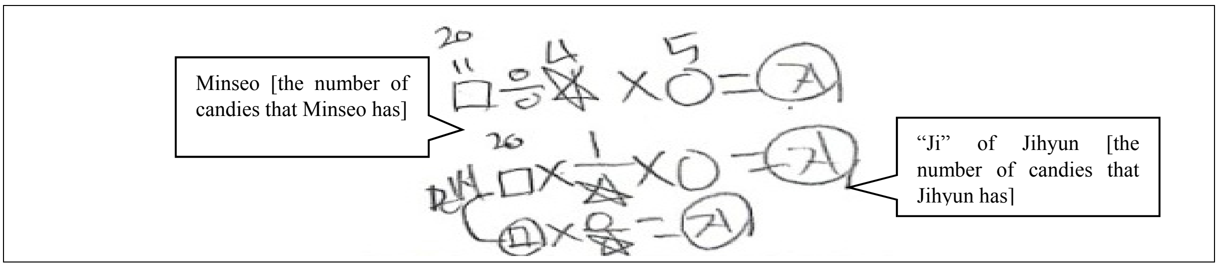


Figure 5. Another example of a sixth grader using variables

with justifying the generalization. This result was a unique feature of sixth graders because fifth graders reasoned the problem structure but had difficulties with representing the equation using variables. More specifically, one of eight sixth graders who did not reason the structure of reverse fraction problems did not verbally generalize the solution method. The other seven represented the generalization, using statements such as "The problem asked about the base quantity, so we need to divide quantity compared by rate." However, these students did not understand the problem structure, so they had difficulties when asked to justify their solution methods. For example, when the interviewer asked how they solved the problems, most of them responded that they just used the algorithm of (quantity compared) ÷ (rate) = (base quantity). Some responded, "The problem did not ask how many times but asked $\frac{2}{3}$ of some, so I thought that it is division." Because these seven students represented the algorithm in words based on memorization instead of generalizing the problem structure or relationship, we scored them as medium in generalization-representation (verbal). They did not justify their problem solution method. The following *Episode 3* illustrates a case in which a sixth grader accurately represented the equation using variables (as shown in *Figure 4*) but did not justify the solution method.

Episode 3. An example of a sixth grader who represented the equation using variables but did not justify it

Interviewer: Can you explain how you solve problem number 1? (See P-1 in *Table 1*)
Student 3: As I learned in school, it looks like this is the base quantity. 10 is the quantity compared and $\frac{2}{3}$ is rate, so I divided 10 by $\frac{2}{3}$ and used the reciprocal. I changed the division to multiplication and simplified it.
Interviewer: How did you solve problem number 4? (See P-4 in *Table 1*)
Student 3: I solved the problem using the same method. I divided quantity compared by rate and got base quantity.
Interviewer: Can you solve this problem using other methods?
Student 3: (long pause) Um… I don't know.

The student in the *Episode 3* explained her solution method, emphasizing "I learned in school." Also, she recognized what looked like the base quantity as well as what would be the quantity compared and rate, respectively. The student mentioned the memorized algorithm step by step and repeatedly used it in solving the reverse fraction problems. However, the student could not come up with another solution method other than the algorithm of (quantity compared) ÷ (rate) = (base quantity). Her struggles mainly arose because she was unable to reason the problem structure, which prevented her from understanding the equivalent relationship between the partial quantity and the fraction representing the partial quantity and from regarding the whole quantity as $\frac{n}{n} = 1$. Note that this outcome was the most frequent case among the sixth graders (n=6). It is likely that many sixth graders learned fraction division without necessarily reasoning the problem structure and simply used the algorithm to solve problems similar to reverse fraction problems. Because the students did not reason the structure of reverse fraction problems, they did not justify their solution method. However, they could represent the equation using variables because they memorized the algorithm introduced in the textbook. On the contrary, sixth graders who reasoned the structure of reverse fraction problems could generalize the solution method, represent, and justify it. *Episode 4* illustrates one of the four students who excelled in all four aspects of the algebraic thinking process.

Episode 4. An example of a sixth grader justifying the generalized solution method

Interviewer: Can you explain the similarities between the solution methods you solved so far?
Student 4: Umm, I divided the original amount [the amount compared] by rate.
Interviewer: You solved it in another way! Can you explain the similarities of such solution methods?
Student 4: I found out one part of the whole and then found out the whole by multiplying.
Interviewer: What do you mean by finding out one part of the whole?
Student 4: If it's $\frac{2}{3}$, I found out $\frac{1}{3}$ first. If it's $\frac{7}{6}$, I found out $\frac{1}{6}$.

Interviewer: Ah, so you found out the quantity of unit fraction? And then?

Student 4: If it's $\frac{1}{3}$, I multiplied it by 3. If it's $\frac{1}{6}$, I multiplied it by 6.

Interviewer: If so, can you explain the similarity among the methods?

Student 4: I divided by the numerator and multiplied by the denominator

Because these students not only used the advanced multiplicative method but also reasoned the structure of reverse fraction problems, they could generalize the solution method, represent it both in words and variables, and justify it. Sixth graders who did not reason the problem structure and solved the problems by simply using the fraction division algorithm did not perform better than fifth graders who were not taught the algorithm. These findings imply that it is important for students to reason the problem structure rather than learn the algorithm to develop algebraic thinking.

DISCUSSIONS AND IMPLICATION

The purpose of this study was to analyze students' algebraic thinking processes in solving reverse fraction problems. We assessed the students' processes in terms of the following aspects of algebraic thinking: problem structure reasoning, generalization-representation, and justification. The findings have implications for teaching algebraic thinking to elementary students.

First, to improve algebraic thinking, a teacher needs to help students reason the structure of problems so that they can comprehend algebraic thinking's features, such as understanding equivalence and transformation using equivalence (Pearn & Stephens, 2018). Because the fifth graders in this study reasoned the structure of reverse fraction problems, they could use algebraic thinking processes — such as generalizing, representing, and justifying — to solve the problems even though they had yet to have lessons on fraction division. On the contrary, the sixth graders who did not reason the problem structure had difficulties justifying their solution methods. Considering that sixth graders who reasoned the problem structure were algebraically proficient, the findings indicate that it is crucial to give students rich opportunities to reason the meanings of problems that they solve in typical mathematics lessons and to discover the mathematical structure behind such problems in order to foster their algebraic thinking processes. The new textbook for Grade 6 in Korea aims for students to understand the calculation principle of fraction division across different problem contexts before formalizing the calculation method (MoE, 2019). Given this shift in the instructional approach, we support a follow-up to this study to see how sixth-grade students who learn fraction division with the new textbooks will solve reverse fraction problems and how proficient they will be in algebraic thinking.

Second, it is important to examine whether students attend to the mathematical structure when representing generalization. The sixth graders in this study outperformed the fifth graders in generalizing the solution method and representing the equation using variables. However, most of the sixth-grade students did not justify their generalizations because they represented it by memorizing the algorithm rather than generalizing the problem structure. In the contrast, the students who paid attention to the problem structure could generalize and justify the solution method. Although fifth graders had some difficulties representing the equation using variables, they used some variables meaningfully in the generalization process. Thus, we need to help students identify, generalize, and represent the mathematical structure when they begin to learn operations (Kieran, 2018). Also, mathematics textbooks and teachers' guidebooks need to be more explicit in exploring the structure and relations behind many mathematics tasks so that students have rich experiences with the generalization process (Pang & S. Kim, 2018).

Third, to improve their algebraic thinking, students should learn to justify their thinking. Even if students can solve a problem and generalize it, they may struggle to actually reason the problem structure without going through the justification process (Blanton et al., 2011) and developing a sophisticated understanding of the relationships between the equation's components. Because Pearn and Stephens (2018) did not assess students' abilities to justify their solution methods, they considered the generalized solution method as evidence of algebraic thinking. However, in this current study, we showed that even the students who generalized the solution method and represented it with variables had difficulties justifying their methods because they were unable to reason the problem structure. Instead of noticing the common problem structure, these students solved the problems only by algorithm, notably the advanced multiplicative method. One way to support these students in advancing their understanding of problem structures is to encourage them to provide justifications for their solutions (Barnett-Clarke, Fisher, Marks, & Ross, 2010). After students solve a problem, we need to ask follow-up questions, such as these: "Can you explain how you solved the problem?" "Why did you do this?" "What do you mean by this?" These types of questions help students reason the problem structure (see *Episode* 2 and *Episode 4*). In such discussions, students can participate in meaningful algebraic thinking processes.

Lastly, in addition to a paper-and-pencil test, cognitive interviews are useful when researchers examine students' algebraic thinking processes. In previous studies conducted by Pang and Cho (2019a, 2019b), the paper-and-pencil test showed that students who used the multiplicative method or the advanced multiplicative method tended to use the same method regardless of problem contexts. Were they to be evaluated solely based on their use of a generalized method, as students were in Pearn and Stephens' study (2018), these students would be classified as thinking more algebraically than other students who had used different solution methods. However, the findings from cognitive interviews in this current study showed differences in students' algebraic thinking processes even though they used the same solution methods in the paper-and-pencil test (i.e., the multiplicative method for the fifth graders and the advanced multiplicative method for the sixth graders). Even the students who used low-level solution methods in the paper-and-pencil test performed well algebraically (see *Table 6* and *Table 7*). These findings suggest that a student's ability to reason the problem structure has more impact on algebraic thinking than the solution method used in the paper-and-pencil test. Thus, to analyze students' algebraic thinking in multiple ways, we need to encourage students to explain their thinking using cognitive interviews as well as paper-and-pencil tests.

In summary, this study's fifth graders, who had not received formal lessons on fraction division, revealed their algebraic thinking mainly by reasoning the problem structure. In contrast, the sixth graders had been taught the division of fractions. They solved problems using the highest-level solution method (i.e., advanced multiplicative method) but did not demonstrate proficient algebraic thinking, likely because their reliance on the fraction division algorithm stunted their reasoning of the problem structure. The findings suggest that teaching fraction operation should include lessons that challenge students to reason the problem structure and relationships among quantities, generalize the common solution methods, represent them in words and variables, and justify their thinking logically.

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